RELATIVE CELLULAR ALGEBRAS

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ABSTRACT. In this paper we generalize cellular algebras by allowing different partial orderings relative to fixed idempotents. For these relative cellular algebras the construction and classification of simples still works similarly as for cellular algebras, but they are e.g. homologically quite different.

We give several examples of algebras which are relative cellular, but not cellular. Most prominently, an annular version of arc algebras, and the restricted enveloping algebra and the small quantum group for $\mathfrak{sl}_2$.

CONTENTS

1. INTRODUCTION

In pioneering work [GL96] Graham–Lehrer introduced the notion of a *cellular algebra*, i.e. an algebra equipped with a so-called *cell datum*. For example, of key importance for this paper, the cell datum comes with a set X and a partial order $<$ on it.

The cell datum provides a method to systematically reduce hard questions about the representation theory of such algebras to problems in linear algebra. Hereby the partial order $<$ on X plays an important role since it yields an "upper triangular way" to construct certain modules, called *cell modules*, which have a crucial role in the theory.

In well-behaved cases the linear algebra problems can be solved giving e.g. a parametrization of the isomorphism classes of simple modules via a subset of X, and a construction of a representative for each class.

In this paper we (strictly) generalize the notion of a cellular algebra to what we call a *relative cellular algebra*, i.e. an algebra equipped with a *relative cell datum*. For example, the relative cell datum comes with a set X, but now with several partial orders $<_{\varepsilon}$ on it, one for each idempotent ε from a preselected set of idempotents. Taking only one idempotent $\varepsilon = 1$, namely the unit, and only one partial order $<_1 = <$, we recover the setting of Graham–Lehrer.

Surprisingly, most of the theory of cellular algebras still works in this relative setup. However, with fairly different proofs, carefully incorporating the various partial orders. The purpose of our paper is to explain this in detail.

Along the way we give examples of algebras which are relative cellular, but not cellular in the sense of Graham–Lehrer.

The papers content in a nutshell. Our exposition follows closely [GL96].

Section 2 In this section we introduce our generalization of cellularity. The crucial new ingredient hereby is (2.1.c) asking for a set E of idempotents and partial orders $<_{\varepsilon}$ for each $\varepsilon \in \mathsf{E}$. Then we define cell modules in our context, and discuss a basis free version of relative cellularity. Last, in Section 2E, we give some first non-trivial examples of relative cellular algebras which are not cellular in the usual sense.

Section 3 This section is the main technical heart of the paper where we recover relative versions of some of the facts which hold for cellular algebras. Most prominently, the construction and classification of simple modules in Theorem 3.17, and some reciprocity laws in Section 3E.

Section 4 In the fourth section we show that restricted enveloping algebra of $\mathfrak{sl}_2$ in positive characteristic are relative cellular algebras. We recover the whole (well-known, of course) representation theory of these algebras from the general theory of relative cellular algebras. We note that the case of the small quantum groups for $\mathfrak{sl}_2$ at roots of unity works mutatis mutandis, giving very similar statements.

Section 5 In the last section we discuss another, and in some sense the motivating, example with respect to relative cellularity: an annular version of arc algebras. We think of this section as being interesting in its own right since annular arc algebras have potential connections to e.g. homological knot theory, exotic t-structures, Springer fibers and modular representation theory.

Moreover, we tried to make the paper reasonably self-contained, and we tried to keep the exposition as easy as possible. In fact, throughout the text we have included several remarks about potential further directions.

Conventions used throughout. We work over any field $\mathbb{K}$ and algebras, maps etc. are assumed to be over $\mathbb{K}$, $\mathbb{K}$-linear etc., and $\otimes = \otimes_{\mathbb{K}}$. Moreover, if not stated otherwise we work with finite-dimensional, left modules. (Even for potentially infinite-dimensional algebras.) By an idempotent ε we always understand a non-zero element in some algebra $\mathfrak{A}$ with $\varepsilon^2 = \varepsilon$.

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2. Relative cellularity

2A. **A generalization of cellularity.** Following [GL96] we define:

Definition 2.1. A *relative cellular algebra* is an associative, unital algebra $\mathfrak{R}$ together with a *(relative) cell datum*, i.e.

$$(\mathsf{X}, \mathsf{M}, \mathsf{C}, {}^{\star}, \mathsf{E}, \mathsf{O}, \epsilon) \tag{2-cell-datum}$$

such that the following hold.

(2.1.a) X is a set and $\mathsf{M} = \{\mathsf{M}(\lambda) \mid \lambda \in \mathsf{X}\}$ a collection of finite sets such that

$$\mathsf{C}\colon \textstyle\coprod_{\lambda\in\mathsf{X}} \mathsf{M}(\lambda) \times \mathsf{M}(\lambda) \to \mathfrak{R}$$

is an injective map with image forming a basis of $\mathfrak{R}$. For $S, T \in \mathsf{M}(\lambda)$ we write $\mathsf{C}(S,T) = \mathsf{C}^{\lambda}_{S,T}$ from now on.

(2.1.b) $\star$ is an anti-involution $\star\colon \mathfrak{R} \to \mathfrak{R}$ such that $(\mathsf{C}^{\lambda}_{S,T})^{\star} = \mathsf{C}^{\lambda}_{T,S}$.

(2.1.c) E is a set of pairwise orthogonal idempotents, all fixed by $\star$, i.e. $\varepsilon^{\star} = \varepsilon$ for all $\varepsilon \in \mathsf{E}$. Further, $\mathsf{O} = \{<_{\varepsilon} \mid \varepsilon \in \mathsf{E}\}$ is a set of partial orders $<_{\varepsilon}$ on X, and ϵ is a map $\epsilon\colon \coprod_{\lambda\in\mathsf{X}} \mathsf{M}(\lambda) \to \mathsf{E}$ sending S to $\epsilon(S) = \varepsilon_S$ such that

$$\text{(2.1-c1)} \qquad \varepsilon\mathfrak{R}\varepsilon\, \mathsf{C}^{\lambda}_{S,T} \in \mathfrak{R}(\leq_{\varepsilon} \lambda), \qquad\qquad \text{(2.1-c2)} \qquad \varepsilon\mathsf{C}^{\lambda}_{S,T} = \begin{cases} \mathsf{C}^{\lambda}_{S,T}, & \text{if } \varepsilon_S = \varepsilon, \\ 0, & \text{if } \varepsilon_S \neq \varepsilon, \end{cases}$$

for all $\lambda \in \mathsf{X}$, $S, T \in \mathsf{M}(\lambda)$ and $\varepsilon \in \mathsf{E}$.

(2.1.d) For $\lambda \in \mathsf{X}$, $S, T \in \mathsf{M}(\lambda)$ and $a \in \mathfrak{R}$ we have

$$\text{(2-left)} \qquad a\mathsf{C}^{\lambda}_{S,T} \in \textstyle\sum_{S'\in\mathsf{M}(\lambda)} r_a(S', S)\, \mathsf{C}^{\lambda}_{S',T} + \mathfrak{R}(<_{\varepsilon_T} \lambda)\varepsilon_T,$$

with scalars $r_a(S', S) \in \mathbb{K}$ only depending on a, S, S'.

We call the set $\{\mathsf{C}^{\lambda}_{S,T} \mid \lambda \in \mathsf{X}, S, T \in \mathsf{M}(\lambda)\}$ a *relative cellular basis.* ▲

In Definition 2.1 we already made use of a notation that will appear throughout the paper: for a subset $\mathsf{I} \subset \mathsf{X}$ we fix the linear subspace

$$\text{(2-1)} \qquad \mathfrak{R}(\mathsf{I}) = \mathbb{K}\{\mathsf{C}^{\lambda}_{S,T} \mid \lambda \in \mathsf{I}, S, T \in \mathsf{M}(\lambda)\} \subset \mathfrak{R}.$$

Often these subspaces will be defined with respect to $<_{\varepsilon}$, for this we abuse notation and $\mathfrak{R}(<_{\varepsilon} \lambda)$ should be understood as $\mathfrak{R}(\{\mu \in \mathsf{X} \mid \mu <_{\varepsilon} \lambda\})$ and similar for analogous expressions. Further, by an *ideal* I *in the poset* $(\mathsf{X}, <_{\varepsilon})$, $<_{\varepsilon}$*-ideal* for short, we understand a subset of $\emptyset \neq \mathrm{I} \subset \mathsf{X}$ such that I is a directed, lower set in the order-theoretical sense. (For example, $<_{\varepsilon} \lambda = \{\mu \in \mathsf{X} \mid \mu <_{\varepsilon} \lambda\}$ is an $<_{\varepsilon}$-ideal.)

Further directions 2.2. We could also work more generally over rings instead of the field $\mathbb{K}$, as e.g. Graham–Lehrer [GL96]. This could be useful to extend the notion of relative cellularity to some affine setup as in [KX12]. However, most of the results in Section 3 use the fact that we work over a field. So, for convenience, we decided not to do so. ▲

The first examples of relative cellular algebras are cellular algebras in the sense of [GL96, Definition 1.1], which we call *usual* for distinction.

Example 2.3. A usual cellular algebra $\mathfrak{U}$ is relative cellular: By construction, (2.1.a) and (2.1.b) are part of the usual cell datum $(\mathsf{X}, \mathsf{M}, \mathsf{C}, \star)$. Next, the set E for (2.1.c) can be taken to be $\mathsf{E} = \{1\}$ (with 1 being the unit of $\mathfrak{R}$) satisfying $1^{\star} = 1$. The partial ordering $<$ of $\mathfrak{U}$ is the partial ordering $<_1$ for the unit. Note hereby that (2.1-c1) follows from (2.1.d) (which is just part of the usual cell datum), while (2.1-c2) is automatic. ▲

Remark 2.4. As we have seen in Example 2.3, the two conditions (2.1-c1) and (2.1-c2) are "invisible" in the usual setup. However, they are crucial for our purposes e.g. (2.1-c1) is used in Lemma 3.12 which in turn gives Theorem 3.17. ▲

As in the usual setup, a relative cellular datum is not unique. Nevertheless, we say that an algebra $\mathfrak{R}$ *is relative cellular* if there exist some relative cellular datum. (Similarly, if we have already fixed part of the relative cell datum as e.g. the anti-involution $\star$.)

Remark 2.5. Note that $|\mathsf{X}| < \infty$ – by (2.1.a) – implies that $\mathfrak{R}$ is finite-dimensional. We will need this assumption sometimes, for example in Theorem 3.17, and we will stress whenever we assume X to be finite. ▲

If not stated otherwise, fix a relative cellular algebra $\mathfrak{R}$ in the following.

2B. **Basic properties.** The (very basic) statements below will be crucial for the definition of cell modules.

Lemma 2.6. The following properties hold.

(2.6.a) For $\lambda \in \mathsf{X}$, $S, T \in \mathsf{M}(\lambda)$, and $\varepsilon \in \mathsf{E}$ we have

$$\mathsf{c}^{\lambda}_{S,T}\, \varepsilon\mathfrak{R}\varepsilon \in \mathfrak{R}(\leq_{\varepsilon} \lambda), \qquad \mathsf{c}^{\lambda}_{S,T}\varepsilon = \begin{cases} \mathsf{c}^{\lambda}_{S,T}, & \text{if } \varepsilon_T = \varepsilon, \\ 0, & \text{if } \varepsilon_T \neq \varepsilon. \end{cases}$$

(2.6.b) For $\varepsilon \in \mathsf{E}$ and a subset $\mathsf{I} \subset \mathsf{X}$ it holds $\varepsilon\mathfrak{R}(\mathsf{I}) \subset \mathfrak{R}(\mathsf{I}) \supset \mathfrak{R}(\mathsf{I})\varepsilon$.

(2.6.c) For an $<_{\varepsilon}$-ideal I_{ε} we have that $\mathfrak{R}(\mathrm{I}_{\varepsilon})\varepsilon$ is a left and $\varepsilon\mathfrak{R}(\mathrm{I}_{\varepsilon})$ is a right ideal in $\mathfrak{R}$.

(2.6.d) For $\lambda \in \mathsf{X}$, $S, T \in \mathsf{M}(\lambda)$, and $a \in \mathfrak{R}$ we have

(2-right) $$\mathsf{c}^{\lambda}_{S,T} a \in \textstyle\sum_{T' \in \mathsf{M}(\lambda)} r_{a^{\star}}(T', T)\, \mathsf{c}^{\lambda}_{S,T'} + \varepsilon_S \mathfrak{R}(<_{\varepsilon_S} \lambda),$$

with the same scalars $r_{a^{\star}}(T', T)$ as in (2.1.d). □

Proof. (2.6.a). This follows by applying $\star$ to (2.1.c).

(2.6.b). The first inclusion follows from (2.1.c) and the second by applying $\star$.

(2.6.c). For the left-ideal-statement let $\mathsf{c}^{\lambda}_{S,T} \in \mathfrak{R}(\mathrm{I}_{\varepsilon})\varepsilon$. Then – by (2.1.d) – we have

$$a\mathsf{c}^{\lambda}_{S,T}\varepsilon \in \textstyle\sum_{S' \in \mathsf{M}(\lambda)} r_a(S', S)\, \mathsf{c}^{\lambda}_{S',T}\varepsilon + \mathfrak{R}(<_{\varepsilon_T} \lambda)\varepsilon_T\varepsilon.$$

But either $\varepsilon_T\varepsilon = 0$ or they agree and the last term is inside the linear subspace. The right-ideal-statement is again obtained using $\star$.

(2.6.d). By applying $\star$ directly to (2.1.d). ■

Combining (2.1.c) and (2.6.a) we obtain:

Corollary 2.7. Let $a \in \mathfrak{R}$ such that $\varepsilon a = a = a\varepsilon$ for $\varepsilon \in \mathsf{E}$. Then

$$a \in \mathbb{K}\{\mathsf{c}^{\lambda}_{S,T} \mid \lambda \in \mathsf{X}, S, T \in \mathsf{M}(\lambda), \varepsilon_S = \varepsilon_T = \varepsilon\}.$$

The same holds for $a^{\star}$ as well. ■

Additionally, Lemma 2.6 gives us a further relation to usual cellular algebras.

Proposition 2.8. Let $\mathfrak{R}$ be a relative cellular algebra with cell datum $(\mathsf{X}, \mathsf{M}, \mathsf{C}, \star, \mathsf{E}, \mathsf{O}, \epsilon)$, and let $\mathfrak{U}$ be a usual cellular algebra with cell datum $(\mathsf{X}, \mathsf{M}, \mathsf{C}, \star)$ and order $<$ on X.

(2.8.a) For all $\varepsilon \in \mathsf{E}$, the algebra $\varepsilon\mathfrak{R}\varepsilon$ is a usual cellular algebra with usual cell datum $(\mathsf{X}, \mathsf{M}_{\varepsilon}, \mathsf{C}_{\varepsilon}, \star)$ and the partial order on X given by $<_{\varepsilon}$,

$$\mathsf{M}_{\varepsilon}(\lambda) = \{S \in \mathsf{M}(\lambda) \mid \varepsilon\mathsf{c}^{\lambda}_{S,T} = \mathsf{c}^{\lambda}_{S,T} \text{ for } T \in \mathsf{M}(\lambda)\},$$

and C_{ε} being the restriction of C to $\coprod_{\lambda \in \mathsf{X}} \mathsf{M}_{\varepsilon}(\lambda) \times \mathsf{M}_{\varepsilon}(\lambda)$.

(2.8.b) The algebra $\mathfrak{U}$ is relative cellular with relative cell datum $(\mathsf{X},\mathsf{M},\mathsf{C},{}^{\star},\{1\},\{<_1\},\epsilon)$, with ϵ mapping everything to 1. $\square$

Proof. *(2.8.a).* That M_{ε} and C_{ε} give a bijection with a basis of $\varepsilon\mathfrak{R}\varepsilon$ follows by combining (2.1.c) and (2.6.a). So we are left with checking the multiplication rule for usual cellular algebras. For $a\in\mathfrak{R}$, $\lambda\in\mathsf{X}$, and $S,T\in\mathsf{M}(\lambda)$ with $\varepsilon_S=\varepsilon_T=\varepsilon$, we use (2.1.c) and get

$$\varepsilon a\varepsilon\,\mathtt{C}^{\lambda}_{S,T}\in\textstyle\sum_{S'\in\mathsf{M}(\lambda),\varepsilon_{S'}=\varepsilon}r_{\varepsilon a\varepsilon}(S',S)\,\mathtt{C}^{\lambda}_{S',T}+\varepsilon\mathfrak{R}(<_{\varepsilon}\lambda)\varepsilon\subset\varepsilon\mathfrak{R}\varepsilon.$$

(2.8.b). This is Example 2.3. ■

Remark 2.9. For any usual cellular algebra $\mathfrak{U}$ and any idempotent ε fixed by * it holds that $\varepsilon\mathfrak{U}\varepsilon$ is cellular in the usual sense, see [KX98, Proposition 4.3]. However, Proposition 2.8 is different since we do not assume $\mathfrak{R}$ to be usual cellular to begin with. ▲

The formal sum $\sum_{\varepsilon\in\mathsf{E}}\varepsilon$, which may be infinite, acts on $\mathfrak{R}$ by left multiplication, since there is exactly one summand that acts non-trivial on each given $\mathtt{C}^{\lambda}_{S,T}$. The proof of the following lemma, showing that E gives a decomposition of the unit of $\mathfrak{R}$, is thus immediate.

Lemma 2.10. The action of $\sum_{\varepsilon\in\mathsf{E}}\varepsilon$ is equal to the identity map of $\mathfrak{R}$. ■

2C. **Existence of cell modules.** We proceed by defining cell modules.

Definition 2.11. For $\lambda\in\mathsf{X}$ and $T\in\mathsf{M}(\lambda)$ let $\Delta(\lambda;T)=\mathbb{K}\{\mathtt{M}^{\lambda}_{S,T}\mid S\in\mathsf{M}(\lambda)\}$. We define an action $\centerdot$ of $\mathfrak{R}$ on $\Delta(\lambda;T)$ by setting

$$a\centerdot\mathtt{M}^{\lambda}_{S,T}=\textstyle\sum_{S'\in\mathsf{M}(\lambda)}r_a(S',S)\,\mathtt{M}^{\lambda}_{S',T},$$

with $r_a(S',S)$ being defined by (2-left). ▲

Lemma 2.12. The action from Definition 2.11 defines the structure of an $\mathfrak{R}$-module on $\Delta(\lambda;T)$. Further, there is an isomorphism of $\mathfrak{R}$-modules $\Delta(\lambda;T)\cong\Delta(\lambda;T')$ for any $T,T'\in\mathsf{M}(\lambda)$. $\square$

Proof. The coefficient $r_a(S',S)$ is – by definition – additive with respect to a, and one has $r_1(S',S)=\delta_{S,S'}$. Moreover, one also has

$$\begin{aligned}a'(a\mathtt{C}^{\lambda}_{S,T})\;&\in\;a'\textstyle\sum_{S'\in\mathsf{M}(\lambda)}r_a(S',S)\,\mathtt{C}^{\lambda}_{S',T}+a'\mathfrak{R}(<_{\varepsilon_T}\lambda)\varepsilon_T\\&\subset\;\textstyle\sum_{S',S''\in\mathsf{M}(\lambda)}r_{a'}(S'',S')r_a(S',S)\,\mathtt{C}^{\lambda}_{S'',T}+\mathfrak{R}(<_{\varepsilon_T}\lambda)\varepsilon_T,\end{aligned}$$

where the inclusion is due to (2-left) and (2.6.c), and

$$(a'a)\mathtt{C}^{\lambda}_{S,T}\in\textstyle\sum_{S''\in\mathsf{M}(\lambda)}r_{a'a}(S'',S)\,\mathtt{C}^{\lambda}_{S'',T}+\mathfrak{R}(<_{\varepsilon_T}\lambda)\varepsilon_T.$$

Thus, we have

$$r_{a'a}(S'',S)=\textstyle\sum_{S'\in\mathsf{M}(\lambda)}r_{a'}(S'',S')r_a(S',S)\;\text{ for }a,a'\in\mathfrak{R}.$$

This in turn implies $a'\centerdot(a\centerdot\mathtt{M}^{\lambda}_{S,T})=(a'a)\centerdot\mathtt{M}^{\lambda}_{S,T}$. Hence, we get a well-defined $\mathfrak{R}$-module structure on $\Delta(\lambda,T)$. Since $r_a(S',S)$ is independent of the second index, the assignment $\mathtt{M}^{\lambda}_{S,T}\mapsto\mathtt{M}^{\lambda}_{S,T'}$ gives an $\mathfrak{R}$-module isomorphism. ■

Due to Lemma 2.12 we omit the T in the definition and notation of $\Delta(\lambda;T)$. We call $\Delta(\lambda)$ a *cell module*, and we denote the basis elements of $\Delta(\lambda)$ by $\mathtt{M}^{\lambda}_{S}$ only. Furthermore – having Lemma 2.12 – we can define right $\mathfrak{R}$-modules:

Definition 2.13. We define the right $\mathfrak{R}$-module $\Delta(\lambda)^{\star}$ on the same vector space as $\Delta(\lambda)$ by setting $\mathtt{M}_S^{\lambda} \centerdot a = a^{\star} \centerdot \mathtt{M}_S^{\lambda}$. ▲

We get – by construction – the following identification:.

Lemma 2.14. The linear extension of the assignment

$$\Theta^{\lambda}\colon \Delta(\lambda) \otimes \Delta(\lambda)^{\star} \to \mathfrak{R}(\{\lambda\}),\ \Theta^{\lambda}(\mathtt{M}_S^{\lambda}, \mathtt{M}_T^{\lambda}) = \mathtt{C}_{S,T}^{\lambda},$$

is an isomorphism of vector spaces. ■

2D. **A basis free definition of relative cellularity.** In this section, we let $\mathfrak{A}$ be an algebra with a fixed anti-involution * and a set E of pairwise orthogonal idempotents, all fixed by *. Furthermore, denote by $\mathbb{K}[\mathsf{E}]$ the semigroup algebra generated by the elements of E. Following [KX98, Definition 3.2] we define:

Definition 2.15. Let $\mathsf{J} \subset \mathfrak{A}$ denote a linear subspace, and let Δ denote a finite-dimensional, left $\mathfrak{A}$-module. Assume that the following hold:

(2.15.a) The linear subspace J is fixed under *, i.e. $\mathsf{J}^{\star} = \mathsf{J}$.

(2.15.b) The linear subspace J is a $\mathbb{K}[\mathsf{E}]$-bimodule.

(2.15.c) There is a $\mathbb{K}[\mathsf{E}]$-bimodule isomorphism $\Theta^{-1}\colon \mathsf{J} \xrightarrow{\cong} \Delta \otimes \Delta^{\star}$ and a diagram

$$\begin{tikzcd}
\mathsf{J} \arrow[r, "\Theta^{-1}"] \arrow[d, "\star"'] \arrow[dr, phantom, "\circlearrowleft"] & \Delta \otimes \Delta^{\star} \arrow[d, "x\otimes y \mapsto y \otimes x"] \\
\mathsf{J} \arrow[r, "(\Theta^{-1})^{\star}"'] & \Delta \otimes \Delta^{\star},
\end{tikzcd}$$

where $\Delta^{\star}$ is the right $\mathfrak{A}$-module on the same vector space as Δ and right action of $\mathfrak{A}$ defined via $x \centerdot a = a^{\star} \centerdot x$.

Then we call J a *cell space*. ▲

Proposition 2.16. A finite-dimensional algebra $\mathfrak{A}$ is relative cellular with respect to * and E if and only if:

(2.16.a) The elements of E give a decomposition of the unit of $\mathfrak{A}$.

(2.16.b) There is some index set X with $|\mathsf{X}| < \infty$ and a vector space decomposition of $\mathfrak{A}$ into cell spaces, i.e. $\mathfrak{A} = \bigoplus_{\lambda \in \mathsf{X}} \mathsf{J}_{\lambda}$.

(2.16.c) For each $\varepsilon \in \mathsf{E}$ there is an enumeration $\mathsf{X} = \{\lambda_1, \lambda_2, \cdots, \lambda_m\}$ such that

$$0 \subset \mathsf{J}_{\lambda_1}^{\oplus}\varepsilon \subset \mathsf{J}_{\lambda_2}^{\oplus}\varepsilon \subset \cdots \subset \mathsf{J}_{\lambda_i}^{\oplus}\varepsilon \subset \cdots \subset \mathsf{J}_{\lambda_m}^{\oplus}\varepsilon \subset \mathfrak{A}\varepsilon, \tag{2-ccc}$$

is a chain of submodules $\mathsf{J}_{\lambda_i}^{\oplus}\varepsilon = \bigoplus_{j=1}^{i} \mathsf{J}_{\lambda_i}\varepsilon$.

(2.16.d) The submodule $\mathsf{J}_{\lambda_i}^{\oplus}\varepsilon$ as in (2-ccc) is a right $\varepsilon'\mathfrak{A}\varepsilon'$-module for any $\varepsilon' \in \mathsf{E}$. □

Proof. Definition 2.1⇒Proposition 2.16. Fix $\varepsilon \in \mathsf{E}$. Since $<_{\varepsilon}$ is a partial order on X, we can inductively construct the linear subspaces $\mathsf{J}_{\lambda_i}\varepsilon \subset \mathfrak{A}\varepsilon$ by starting with

$$\mathsf{J}_{\lambda_1}\varepsilon = \mathbb{K}\{\mathtt{C}_{S,T}^{\lambda_1} \mid S,T \in \mathsf{M}(\lambda_1), \varepsilon_T = \varepsilon\} \overset{(2.1\text{-c2})}{=} \mathbb{K}\{\mathtt{C}_{S,T}^{\lambda_1}\varepsilon \mid S,T \in \mathsf{M}(\lambda_1), \varepsilon_T = \varepsilon\}$$

for some $<_{\varepsilon}$-minimal $\lambda_1 \in \mathsf{X}$. Then we set $\mathsf{J}_{\lambda_i}^{\oplus}\varepsilon = \bigoplus_{j=1}^{i} \mathsf{J}_{\lambda_i}\varepsilon$, and the so constructed linear spaces are submodules and satisfy the cell chain condition (2-ccc) by (2.1.d). Moreover, orthogonality and the *-version of (2.1-c1) (see (2.6.a)) shows that (2.16.d) holds as well.

Further, define $\mathsf{J}_\lambda = \bigoplus_{\varepsilon\in\mathsf{E}} \mathsf{J}_\lambda\varepsilon$. These are cell spaces: By (2.1.b) and the fact that $\varepsilon_S = \varepsilon$ for some $\varepsilon \in \mathsf{E}$ we get (2.15.a), while (2.15.b) follows from (2.1-c2). Next – by virtue of construction – $\mathsf{J}_\lambda = \mathfrak{A}(\{\lambda\})$. Thus, we can set $\Delta_\lambda \cong \Delta(\lambda)$, whose properties – by Lemma 2.14 – give (2.15.c) by defining $\Theta^{-1}(\mathtt{C}^\lambda_{S,T}) = (\mathtt{M}^\lambda_S, \mathtt{M}^\lambda_T)$. Last – by (2.1.a), Lemma 2.10 and finite-dimensionality – we get (2.16.a) and (2.16.b).

Proposition 2.16⇒Definition 2.1. First, let $\mathsf{X} = \{\lambda \mid \mathsf{J}_\lambda \text{ is a cell space}\}$. For any cell space J_λ we first fix a basis $\{\mathtt{M}^\lambda_S\}$ of its associated Δ_λ. Note that – by finite-dimensionality – we can choose this to be a basis consisting of common eigenvectors for $\mathbb{K}[\mathsf{E}]$, and we thus can demand that this basis satisfies either $\varepsilon\mathtt{M}^\lambda_S = \mathtt{M}^\lambda_S$ or $\varepsilon\mathtt{M}^\lambda_S = 0$ for each $\varepsilon \in \mathsf{E}$. The λ, S, T play hereby the role of some indexes, where we set $\mathsf{M}(\lambda)$ to be the set of all S,T's which appear in this enumeration. Next, use (2.15.c) to define $\mathsf{C}(S,T) = \mathtt{C}^\lambda_{S,T} = \Theta^{-1}(\mathtt{M}^\lambda_S \otimes \mathtt{M}^\lambda_T)$ for $S,T \in \mathsf{M}(\lambda)$. Since we have already fixed $\star$, this defines the relative cell datum up to the part about idempotents. To define the remaining data, first note that E is already given. Moreover, the cell chain condition (2-ccc) gives rise to a partial ordering $<_\varepsilon$ on X for each $\varepsilon \in \mathsf{E}$. Last, observe that $\varepsilon(S)\mathtt{C}^\lambda_{S,T} = \mathtt{C}^\lambda_{S,T}$ for precisely one $\varepsilon(S) \in \mathsf{E}$ due to the choice of the basis $\{\mathtt{M}^\lambda_S\}$, orthogonality and (2.16.a). Thus, we can define $\varepsilon_S = \varepsilon(S)$, which gives us the last part of the relative cell datum.

It remains to check that we have defined a relative cell datum. First, note that all $\mathsf{M}(\lambda)$'s are finite because – by assumption – the Δ_λ's are finite-dimensional, while $|\mathsf{X}| < \infty$ – also by assumption. Second – by (2.16.b) – we have an isomorphism of vector spaces $\mathfrak{R} \cong \bigoplus_{\lambda\in\mathsf{X}} \mathsf{J}_\lambda$, which shows that (2.1.a) holds. That (2.1.b) holds follows from the commutative diagram in (2.15.c), while (2.1.d) follows from (2.15.b). Last, it remains to show (2.1-c1) and (2.1-c2), where the latter is clear by construction of ϵ. The remaining part follows then by applying $\star$ to (2.16.d). ■

Further directions 2.17. As explained in [KX98], the basis free formulation of usual cellularity is connected to ideals in the setting of quasi-hereditary algebras. In the relative setup we lose the ideal structure (cf. (2.16.c) and (2.16.d)) and we do not know what the relative version of the connection to quasi-hereditary algebras is. ▲

2E. Examples of relative cellular algebras.

Remark 2.18. For the following examples recall that the *Cartan matrix* $\mathcal{C}(\mathfrak{A})$ of some finite-dimensional algebra $\mathfrak{A}$ is defined by counting the multiplicities of the simples L in the indecomposable projectives P. Now, it follows from [KX99, Proposition 3.2] that $\mathcal{C}(\mathfrak{U})$ is symmetric and positive definite in case $\mathfrak{U}$ is a usual cellular algebra. ▲

Example 2.19. Consider the type A_n graphs with doubled edges (where we exclude the case $n = 2$ because it requires a slightly different setup):

$$\mathrm{A}_n = \mathtt{1} \rightleftarrows \mathtt{2} \rightleftarrows \mathtt{3} \rightleftarrows \cdots \rightleftarrows \mathtt{n}\ , \quad (\mathtt{i}{\to}\mathtt{j})^\star = \mathtt{j}{\to}\mathtt{i},$$

(2-2)

$$\text{Relations:}\quad \begin{array}{l} \text{The loops of length two are all equal, i.e. } \mathtt{i}{\to}\mathtt{j}{\to}\mathtt{i} = \mathtt{i}{\to}\mathtt{k}{\to}\mathtt{i}; \\ \text{Going three steps in one direction is zero, e.g. } \mathtt{1}{\to}\mathtt{2}{\to}\mathtt{3} = 0. \end{array}$$

We let $\mathfrak{U}(\mathrm{A}_n)$ be the quotient of the path algebra of A_n (multiplication $\circ$ being composition of paths $\mathtt{i}{\to}\mathtt{j} \circ \mathtt{j}{\to}\mathtt{k} = \mathtt{i}{\to}\mathtt{j}{\to}\mathtt{k}$) with relations as in (2-2). Up to base change one gets:

$$\mathcal{C}(\mathfrak{U}(\mathrm{A}_3)) = \begin{pmatrix} 2&1&0\\1&2&1\\0&1&2 \end{pmatrix},\quad \mathcal{C}(\mathfrak{U}(\mathrm{A}_4)) = \begin{pmatrix} 2&1&0&0\\1&2&1&0\\0&1&2&1\\0&0&1&2 \end{pmatrix},\quad \mathcal{C}(\mathfrak{U}(\mathrm{A}_5)) = \begin{pmatrix} 2&1&0&0&0\\1&2&1&0&0\\0&1&2&1&0\\0&0&1&2&1\\0&0&0&1&2 \end{pmatrix},\quad \text{etc.},$$

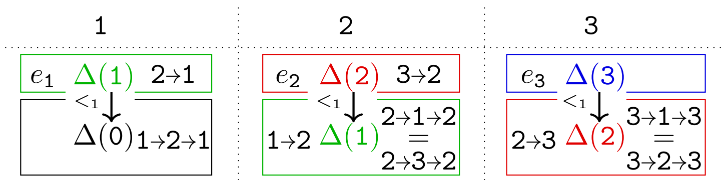
all of which are positive definite. The algebra $\mathfrak{U}(\mathrm{A}_n)$ is known as the *type* A_n *zig-zag algebra*, cf. [HK01, Section 3]. Let us discuss the case $n = 2$ with respect to cellularity in detail, the general case works mutatis mutandis.

First, the $\mathfrak{U}(\mathrm{A}_3)$-action on itself is given by pre-composition of paths, and the algebra can be equipped with the anti-involution $\star$ indicated in (2-2) which fixes the vertex idempotents e_1, e_2, e_3. Clearly, $\mathfrak{U}(\mathrm{A}_3)$ has one-dimensional simple modules $L(\mathtt{i})$ for $i \in \{1, 2, 3\}$ on which $e_{\mathtt{j}}$ acts by δ_{ij}.

The algebra $\mathfrak{U}(\mathrm{A}_3)$ is a relative cellular algebra with respect to $\star$. As a relative cell datum we can take

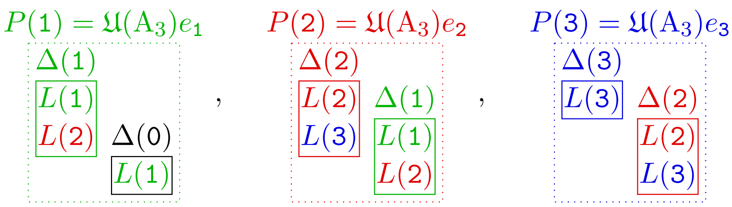
$$\mathsf{X} = \{\mathsf{0} <_1 \mathsf{1} <_1 \mathsf{2} <_1 \mathsf{3}\},$$
$$\mathsf{M}(\mathsf{0}) = \{1{\to}2{\to}1\},\ \ \mathsf{M}(\mathsf{1}) = \{e_1, 2{\to}1\},\ \ \mathsf{M}(\mathsf{2}) = \{e_2, 3{\to}2\},\ \ \mathsf{M}(\mathsf{3}) = \{e_3\},\ \ \mathsf{C}^{\mathtt{i}}_{S,T} = S \circ T^{\star},$$
$$\mathsf{E} = \{1\},\quad \epsilon(1{\to}2{\to}1) = \epsilon(e_1) = \epsilon(2{\to}1) = \epsilon(e_2) = \epsilon(3{\to}2) = \epsilon(e_3) = 1.$$

(We think of $\mathsf{0}$ as playing the role of a dummy.) Note that $\mathsf{E} = \{1\}$ is the same choice as in Example 2.3, and $\mathfrak{U}(\mathrm{A}_3)$ is actually cellular in the usual sense. Now, the cellular basis and cell modules are given as follows, where we write $\mathtt{i}$ on top of the columns containing $\Delta(\lambda; T)$'s with $\varepsilon_T = e_{\mathtt{i}}$ (in the notation from Definition 2.11):



The left action going in the indicated direction (or it stays within the Δ's) as one easily checks. Note the directedness: $\Delta(\mathsf{0}) \xleftarrow{<_1} \Delta(\mathsf{1}) \xleftarrow{<_1} \Delta(\mathsf{2}) \xleftarrow{<_1} \Delta(\mathsf{3})$, making the cell modules well-defined since they are obtained by modding out terms which are $<_1$-smaller.

Further, the indecomposable projectives are



which have the indicated Δ-filtrations. We will see in Proposition 3.19 that this is a general feature, with partial order in the filtration being relative. See also Example 2.20 and Example 2.21 below. ▲

Morally speaking, in the relative setup we can separate parts which are cellular by using the idempotents in E. Here two prototypical examples:

Example 2.20. (We use a notation similar as in Example 2.19.) Consider the following family of quivers, i.e. the cycles on n vertices with double edges:

$$
\tilde{\mathrm{A}}_n = \begin{tikzcd} \mathtt{1} \arrow[r, shift left] \arrow[d, shift left] & \mathtt{2} \arrow[l, shift left] \arrow[d, shift left] \\ \mathtt{n} \arrow[u, shift left] \arrow[r, shift left, dash] & \cdots \arrow[l, shift left] \arrow[r, shift left] & \mathtt{3} \arrow[u, shift left] \arrow[l, shift left] \end{tikzcd}, \quad (\mathtt{i}{\to}\mathtt{j})^\star = \mathtt{j}{\to}\mathtt{i}, \tag{2-3}
$$

$$
\text{Relations:} \begin{array}{l} \text{The loops of length two are all equal, i.e. } \mathtt{i}{\to}\mathtt{j}{\to}\mathtt{i} = \mathtt{i}{\to}\mathtt{k}{\to}\mathtt{i}; \\ \text{Going three steps in one direction is zero, e.g.} \mathtt{1}{\to}\mathtt{n}{\to}\mathtt{n}{-}\mathtt{1}{=}0. \end{array}
$$

(The case $n = 2$ is special and excluded.) As in Example 2.19, we let $\mathfrak{R}(\tilde{\mathrm{A}}_n)$ be the corresponding quotient of the path algebra of $\tilde{\mathrm{A}}_n$, with relations given in (2-3), and anti-involution $\star$ given by swapping the orientations of the arrows. Again, the Cartan matrices are easy to calculate and up to base change:

$$
\mathcal{C}(\mathfrak{R}(\tilde{\mathrm{A}}_3)) = \begin{pmatrix} 2&1&1\\1&2&1\\1&1&2 \end{pmatrix}, \quad \mathcal{C}(\mathfrak{R}(\tilde{\mathrm{A}}_4)) = \begin{pmatrix} 2&1&0&1\\1&2&1&0\\0&1&2&1\\1&0&1&2 \end{pmatrix}, \quad \mathcal{C}(\mathfrak{R}(\tilde{\mathrm{A}}_5)) = \begin{pmatrix} 2&1&0&0&1\\1&2&1&0&0\\0&1&2&1&0\\0&0&1&2&1\\1&0&0&1&2 \end{pmatrix}, \quad \text{etc.}
$$

The algebra $\mathfrak{R}(\tilde{\mathrm{A}}_n)$ is known as the *type* $\tilde{\mathrm{A}}_n$ *zig-zag algebra*, and is for example studied in the context of categorical actions, see e.g. [GTW17, Section 3.1] or [MT16, Section 2.3]. In contrast to $\mathfrak{U}(\mathrm{A}_n)$, the algebra $\mathfrak{R}(\tilde{\mathrm{A}}_n)$ is not cellular in the usual sense (at least for even n where the Cartan matrix is only positive semidefinite), but it is relative cellular as we discuss now in the case $n = 3$, the general case again being similar.

In this case we take the following relative cell datum. Let $\varepsilon = e_2 + e_3$ and let

$$
\mathsf{X} = \{\mathsf{2} <_{e_1} \mathsf{3} <_{e_1} \mathsf{1}\} = \{\mathsf{1} <_{\varepsilon} \mathsf{2} <_{\varepsilon} \mathsf{3}\},
$$
$$
\mathsf{M}(\mathsf{1}) = \{e_1, \mathsf{2}{\to}\mathsf{1}\}, \quad \mathsf{M}(\mathsf{2}) = \{e_2, \mathsf{3}{\to}\mathsf{2}\}, \quad \mathsf{M}(\mathsf{3}) = \{e_3, \mathsf{1}{\to}\mathsf{3}\}, \quad \mathsf{C}^{\mathtt{i}}_{S,T} = S \circ T^\star,
$$
$$
\mathsf{E} = \{e_1, \varepsilon\}, \; \epsilon(e_1) = \epsilon(\mathsf{1}{\to}\mathsf{3}) = e_1, \epsilon(e_2) = \epsilon(e_3) = \epsilon(\mathsf{2}{\to}\mathsf{1}) = \epsilon(\mathsf{3}{\to}\mathsf{2}) = \varepsilon.
$$

Next, the relative cellular basis and the cell modules:

1	2	3
$\boxed{e_1 \;\; \Delta(\mathsf{1}) \;\; \mathsf{2}{\to}\mathsf{1}}$	$\boxed{e_2 \;\; \Delta(\mathsf{2}) \;\; \mathsf{3}{\to}\mathsf{2}}$	$\boxed{e_3 \;\; \Delta(\mathsf{3}) \;\; \mathsf{1}{\to}\mathsf{3}}$
$<_{e_1} \downarrow$	$<_{\varepsilon} \downarrow$	$<_{\varepsilon} \downarrow$
$\boxed{\mathsf{3}{\to}\mathsf{1} \;\; \Delta(\mathsf{3}) \;\; \begin{array}{c}\mathsf{1}{\to}\mathsf{2}{\to}\mathsf{1}\\=\\\mathsf{1}{\to}\mathsf{3}{\to}\mathsf{1}\end{array}}$	$\boxed{\mathsf{1}{\to}\mathsf{2} \;\; \Delta(\mathsf{1}) \;\; \begin{array}{c}\mathsf{2}{\to}\mathsf{1}{\to}\mathsf{2}\\=\\\mathsf{2}{\to}\mathsf{3}{\to}\mathsf{2}\end{array}}$	$\boxed{\mathsf{2}{\to}\mathsf{3} \;\; \Delta(\mathsf{2}) \;\; \begin{array}{c}\mathsf{3}{\to}\mathsf{1}{\to}\mathsf{3}\\=\\\mathsf{3}{\to}\mathsf{2}{\to}\mathsf{3}\end{array}}$

Hereby we like to stress the difference between $\Delta(\mathsf{1})$ in the left and middle column: The one in the left column is $\Delta(\mathsf{1}, e_1)$, the other is $\Delta(\mathsf{1}, \mathsf{2}{\to}\mathsf{1})$, the first of which is defined using the partial order $<_{e_1}$, the second the partial order $<_{\varepsilon}$.

Last, the indecomposable projectives themselves are

$$
P(\mathsf{1}) = \mathfrak{R}(\tilde{\mathrm{A}}_3)e_1 \quad P(\mathsf{2}) = \mathfrak{R}(\tilde{\mathrm{A}}_3)e_2 \quad P(\mathsf{3}) = \mathfrak{R}(\tilde{\mathrm{A}}_3)e_3
$$
$$
\begin{array}{ll} \Delta(\mathsf{1}) & \\ \boxed{\begin{array}{c} L(\mathsf{1}) \\ L(\mathsf{2}) \end{array}} & \begin{array}{c} \Delta(\mathsf{3}) \\ \boxed{\begin{array}{c} L(\mathsf{3}) \\ L(\mathsf{1}) \end{array}} \end{array} \end{array}, \quad
\begin{array}{ll} \Delta(\mathsf{2}) & \\ \boxed{\begin{array}{c} L(\mathsf{2}) \\ L(\mathsf{3}) \end{array}} & \begin{array}{c} \Delta(\mathsf{1}) \\ \boxed{\begin{array}{c} L(\mathsf{1}) \\ L(\mathsf{2}) \end{array}} \end{array} \end{array}, \quad
\begin{array}{ll} \Delta(\mathsf{3}) & \\ \boxed{\begin{array}{c} L(\mathsf{3}) \\ L(\mathsf{1}) \end{array}} & \begin{array}{c} \Delta(\mathsf{2}) \\ \boxed{\begin{array}{c} L(\mathsf{2}) \\ L(\mathsf{3}) \end{array}} \end{array} \end{array}
$$

which have order depended cyclic patterns. ▲

Example 2.21. (We use a notation similar as in Example 2.19.) As in Example 2.20 we use the graphs $\tilde{\mathrm{A}}_n$ to define a quiver algebra $\mathfrak{R}'(\tilde{\mathrm{A}}_n)$. But we impose the relations in (2-4) instead of those in (2-3). (We keep the anti-involution $\star$.)

$$\text{(2-4)}\qquad \text{Relations:}\quad \begin{array}{c}\text{The loops of length two are all equal, i.e. } \mathtt{i}{\to}\mathtt{j}{\to}\mathtt{i}=\mathtt{i}{\to}\mathtt{k}{\to}\mathtt{i};\\ \text{Going around the circle is zero, e.g. } \mathtt{1}{\to}\mathtt{2}{\to}\cdots{\to}\mathtt{n}{\to}\mathtt{1}=0.\end{array}$$

The Cartan matrices are, up to base change, now

$$\mathcal{C}(\mathfrak{R}'(\tilde{\mathrm{A}}_3))=\begin{pmatrix}3&3&3\\3&3&3\\3&3&3\end{pmatrix},\quad \mathcal{C}(\mathfrak{R}'(\tilde{\mathrm{A}}_4))=\begin{pmatrix}4&4&4&4\\4&4&4&4\\4&4&4&4\\4&4&4&4\end{pmatrix},\quad \mathcal{C}(\mathfrak{R}'(\tilde{\mathrm{A}}_5))=\begin{pmatrix}5&5&5&5&5\\5&5&5&5&5\\5&5&5&5&5\\5&5&5&5&5\\5&5&5&5&5\end{pmatrix},\quad \text{etc.,}$$

which are not positive definite giving us that the $\mathfrak{R}'(\tilde{\mathrm{A}}_n)$ are not cellular algebras in the usual sense. However, they are cellular in the relative sense, where we as before discuss the $n=3$ case in detail, the general case being similar. We can take

$$\mathsf{X}=\{3<_{e_1}2<_{e_1}1\}=\{1<_{e_2}3<_{e_2}2\}=\{2<_{e_3}1<_{e_3}3\},$$
$$\mathsf{M}(1)=\{e_1,3{\to}1,2{\to}3{\to}1\},\quad \mathsf{M}(2)=\{e_2,1{\to}2,3{\to}1{\to}2\},\quad \mathsf{M}(3)=\{e_3,2{\to}3,1{\to}2{\to}3\},$$
$$\mathsf{C}^{\mathtt{i}}_{S,T}=S\circ T^{\star},\quad \mathsf{E}=\{e_1,e_2,e_3\},\quad \epsilon(\mathtt{i}{\to}\cdot)=e_{\mathtt{i}}.$$

The relative cellular basis and the cell modules are then

$$\begin{array}{cccc|c|c}
 & & \mathsf{1} & & \mathsf{2} & \mathsf{3}\\
\hline
e_1 & \Delta(1) & 3{\to}1 & 2{\to}3{\to}1 & \Delta(2) & \Delta(3)\\
 & <_{e_1}\downarrow & & & <_{e_2}\downarrow & <_{e_3}\downarrow\\
2{\to}1 & \Delta(2) & \begin{smallmatrix}1{\to}2{\to}1\\=\\1{\to}3{\to}1\end{smallmatrix} & \begin{smallmatrix}3{\to}1{\to}2{\to}1\\=\\3{\to}1{\to}3{\to}1\end{smallmatrix} & \Delta(3) & \Delta(1)\\
 & <_{e_1}\downarrow & & & <_{e_2}\downarrow & <_{e_3}\downarrow\\
3{\to}2{\to}1 & \Delta(3) & \begin{smallmatrix}2{\to}3{\to}2{\to}1\\=\\2{\to}1{\to}2{\to}1\end{smallmatrix} & \begin{smallmatrix}1{\to}2{\to}3{\to}2{\to}1\\=\\1{\to}2{\to}1{\to}2{\to}1\end{smallmatrix} & \Delta(1) & \Delta(2)
\end{array}$$

with the cell modules in the second and third columns being analog.

Last, the indecomposable projectives themselves are

$$\begin{array}{ccc}
\multicolumn{3}{c}{P(1)=\mathfrak{R}'(\tilde{\mathrm{A}}_3)e_1}\\
\Delta(1) & & \\
\boxed{L(1)} & \Delta(2) & \\
\boxed{L(3)} & \boxed{L(2)} & \Delta(3)\\
\boxed{L(2)} & \boxed{L(1)} & \boxed{L(3)}\\
 & \boxed{L(3)} & \boxed{L(2)}\\
 & & \boxed{L(1)}
\end{array},\quad
\begin{array}{ccc}
\multicolumn{3}{c}{P(2)=\mathfrak{R}'(\tilde{\mathrm{A}}_3)e_2}\\
\Delta(2) & & \\
\boxed{L(2)} & \Delta(3) & \\
\boxed{L(1)} & \boxed{L(3)} & \Delta(1)\\
\boxed{L(3)} & \boxed{L(2)} & \boxed{L(1)}\\
 & \boxed{L(1)} & \boxed{L(3)}\\
 & & \boxed{L(2)}
\end{array},\quad
\begin{array}{ccc}
\multicolumn{3}{c}{P(3)=\mathfrak{R}'(\tilde{\mathrm{A}}_3)e_3}\\
\Delta(3) & & \\
\boxed{L(3)} & \Delta(1) & \\
\boxed{L(2)} & \boxed{L(1)} & \Delta(2)\\
\boxed{L(1)} & \boxed{L(3)} & \boxed{L(2)}\\
 & \boxed{L(2)} & \boxed{L(1)}\\
 & & \boxed{L(3)}
\end{array}$$

which again have (quite heavy) cyclic patterns. ▲

Remark 2.22. In the above three examples we leave it to the reader to check that (2.1.a) to (2.1.d) hold. (For Example 2.19: (2.1.d) is the most crucial thing to be checked, with (2.1.c) then being automatic. See also Example 2.3 and Remark 2.4. For Example 2.20: In this case (2.1-c1) needs to be checked. It follows since e.g. $e_1\mathfrak{R}(\tilde{\mathrm{A}}_3)e_1$ equals the linear span of all loops at the vertex $\mathtt{1}$, which are either e_1 or act on everything except e_1 as zero. For Example 2.21: Again, (2.1-c1) is non-trivial. However, it can be checked by keeping in mind that $e_{\mathtt{i}}\mathfrak{R}'(\tilde{\mathrm{A}}_3)e_{\mathtt{i}}$ equals the linear span of all loops at the vertex $\mathtt{i}$.) ▲

Example 2.23. Let $\mathbb{K}$ be a field of positive characteristic $p > 0$. In Section 4 we show that the restricted enveloping algebra $\mathfrak{u}_0(\mathfrak{sl}_2)$ is relative cellular, but not cellular. (Except in case $p = 2$ where $\mathfrak{u}_0(\mathfrak{sl}_2)$ is actually already cellular, see Remark 4.6.)

Similarly, let $\mathbb{K}$ be any field and fix $q \in \mathbb{K}$ to be a root of unity, $q \neq \pm 1$. The case of the so-called small quantum group $\mathfrak{u}_q(\mathfrak{sl}_2)$ at q associated to $\mathfrak{sl}_2$ (see e.g. [Lus90]) works mutatis mutandis as for $\mathfrak{u}_0(\mathfrak{sl}_2)$, i.e. $\mathfrak{u}_q(\mathfrak{sl}_2)$ is relative cellular, but not cellular in the usual sense as long as $q \neq \pm\sqrt{-1}$. ▲

Example 2.24. Another example is an annular version of arc algebras $\mathfrak{A}rc_n^{\mathrm{ann}}$ which we discuss in detail in Section 5. Note that $\mathfrak{A}rc_n^{\mathrm{ann}}$ is again not a usual cellular algebra, but only a relative cellular algebra, cf. Proposition 5.22. ▲

Further directions 2.25. The most famous examples of usual cellular algebras are coming from centralizer algebras as e.g. Hecke, Temperley–Lieb or Brauer algebras. These arise from fairly general constructions via the theory of tilting modules, see e.g. [AST18] or [BT17a, Appendix A]. We do not know what the relative version of this is. ▲

3. Simple and projective modules

In the present section we discuss the representation theory of relative cellular algebras, following [GL96, Sections 2 and 3]. We stress hereby that some of the statements, e.g. Theorem 3.17 and Theorem 3.23, hold verbatim as for usual cellular algebras. However, our proofs here are, and have to be, quite different.

We continue to use the notation from Section 2. In particular, $\mathfrak{R}$ denotes a relative cellular algebra with relative cell datum as in (2-cell-datum).

3A. Simple quotients of cell modules.

First, we define a bilinear form on cell modules to get a better handle on their structure.

Lemma 3.1. Let $\lambda \in \mathsf{X}$ and $a \in \mathfrak{R}$. Then, for $S, T, U, V \in \mathsf{M}(\lambda)$, we have

$$\mathtt{C}_{U,S}^{\lambda}\, a\, \mathtt{C}_{T,V}^{\lambda} \in \phi_a(S,T)\mathtt{C}_{U,V}^{\lambda} + (\varepsilon_U \mathfrak{R}(<_{\varepsilon_U} \lambda) \cap \mathfrak{R}(<_{\varepsilon_V} \lambda)\varepsilon_V)\,,$$

where $\phi_a(S,T) = r_{\mathtt{C}_{U,S}^{\lambda} a}(U,T) = r_{a^{\star}\mathtt{C}_{V,T}^{\lambda}}(V,S) \in \mathbb{K}$. □

Proof. We apply (2-left) respectively (2-right) and compare coefficients. The statement then follows immediately. ■

Thus, by Lemma 3.1, we can define $\phi_a(S,T)$ as therein and it is independent of the choice of U and V in the statement of Lemma 3.1. Of special importance is the case $\phi_1 = \phi_{a=1}$, with 1 being the unit of $\mathfrak{R}$.

Definition 3.2. For $\lambda \in \mathsf{X}$ we define a bilinear form $\phi^{\lambda} \colon \Delta(\lambda) \times \Delta(\lambda) \to \mathbb{K}$ by setting $\phi^{\lambda}(\mathtt{M}_S^{\lambda}, \mathtt{M}_T^{\lambda}) = \phi_1(S,T)$ for $S, T \in \mathsf{M}(\lambda)$, and extending bilinearly. ▲

For (3.3.c) of the following lemma recall Θ^{λ} as defined in Lemma 2.14. Its proof is mutatis mutandis as in [GL96, Proposition 2.4] and omitted.

Lemma 3.3. For $\lambda \in \mathsf{X}$ we have the following.

(3.3.a) The bilinear form ϕ^{λ} is symmetric.

(3.3.b) For $a \in \mathfrak{R}$ and $x, y \in \Delta(\lambda)$ we have $\phi^{\lambda}(a \centerdot x, y) = \phi^{\lambda}(x, a^{\star} \centerdot y)$.

(3.3.c) For $u, x, y \in \Delta(\lambda)$ we have $\Theta^{\lambda}(u \otimes x) \centerdot y = \phi^{\lambda}(x,y)u$. ■

The main use of ϕ^{λ} is Corollary 3.5 below: Elements of $\Delta(\lambda)$ not contained in the radical of ϕ^{λ} are cyclic generators for $\Delta(\lambda)$. Hereby, as usual, the *radical of* ϕ^{λ} is $\mathrm{rad}(\lambda)=\{x\in\Delta(\lambda)\mid\phi^{\lambda}(x,y)=0 \text{ for all } y\in\Delta(\lambda)\}$, which is a linear subspace of $\Delta(\lambda)$.

Lemma 3.4. Let $\lambda\in\mathsf{X}$ and $z\in\Delta(\lambda)$. Then

$$\mathfrak{R}(\{\lambda\})\centerdot z=\mathrm{im}(\phi^{\lambda}(\_,z))\Delta(\lambda)\subset\mathfrak{R}\centerdot z.$$

In particular, if $\mathrm{im}(\phi^{\lambda}(\_,z))=\mathbb{K}$, then we have $\Delta(\lambda)=\mathfrak{R}(\{\lambda\})\centerdot z=\mathfrak{R}\centerdot z$. □

Proof. Let $y\in\Delta(\lambda)$ and $S,T\in\mathsf{M}(\lambda)$. By (3.3.c) we have

$$\mathtt{C}^{\lambda}_{S,T}\centerdot z=\Theta^{\lambda}(\mathtt{M}^{\lambda}_{S}\otimes\mathtt{M}^{\lambda}_{T})\centerdot z=\phi^{\lambda}(\mathtt{M}^{\lambda}_{T},z)\mathtt{M}^{\lambda}_{S}\in\mathrm{im}(\phi^{\lambda}(\_,z))\Delta(\lambda),$$

and reversely

$$\phi^{\lambda}(y,z)\mathtt{M}^{\lambda}_{S}=\Theta^{\lambda}(\mathtt{M}^{\lambda}_{S}\otimes y)\centerdot z\in\mathfrak{R}(\{\lambda\})\centerdot z.$$

Hence, we have equality. The special case is then clear. ■

Since we work over a field we get as a direct consequence:

Corollary 3.5. We have $z\in\Delta(\lambda)\setminus\mathrm{rad}(\lambda)$ if and only if $\mathfrak{R}(\{\lambda\})\centerdot z=\Delta(\lambda)$. ■

Next, $\mathrm{rad}(\lambda)$ allows us to deduce that cell modules have either a trivial or a simple head.

Proposition 3.6. Let $\lambda\in\mathsf{X}$.

(3.6.a) The radical $\mathrm{rad}(\lambda)$ is a submodule of $\Delta(\lambda)$.

(3.6.b) If ϕ^{λ} is non-zero, then $\Delta(\lambda)/\mathrm{rad}(\lambda)$ is simple.

(3.6.c) If ϕ^{λ} is non-zero, then $\Delta(\lambda)/\mathrm{rad}(\lambda)$ is the head of $\Delta(\lambda)$. □

Proof. *(3.6.a).* This follows immediately from (3.3.b).

(3.6.b). By Corollary 3.5, any $z\in\Delta(\lambda)\setminus\mathrm{rad}(\lambda)$ generates $\Delta(\lambda)$. Thus, the claim follows.

(3.6.c). Again by Corollary 3.5, any $z\in\Delta(\lambda)\setminus\mathrm{rad}(\lambda)$ generates $\Delta(\lambda)$. Hence, any proper submodule of $\Delta(\lambda)$ is contained in $\mathrm{rad}(\lambda)$. Thus, $\mathrm{rad}(\lambda)$ is the unique maximal submodule of $\Delta(\lambda)$ and so equal to the (representation theoretical) radical $\mathrm{Rad}(\Delta(\lambda))$. (Recall that $\mathrm{Rad}(\Delta(\lambda))$ is intersection of all proper, maximal submodules of $\Delta(\lambda)$.) ■

We write $\mathsf{X}_0=\{\lambda\in\mathsf{X}\mid\phi^{\lambda}\text{ is non-zero}\}$. Having Proposition 3.6 we can define:

Definition 3.7. For $\lambda\in\mathsf{X}_0$, we set $L(\lambda)=\Delta(\lambda)/\mathrm{rad}(\lambda)$. ▲

3B. **Morphisms between cell modules.** In contrast to the usual setup, the existence of morphisms between cell modules is a less useful tool as we will see.

Lemma 3.8. Let $\lambda\in\mathsf{X}_0$, $\mu\in\mathsf{X}$, and $f\in\mathrm{Hom}_{\mathfrak{R}}(\Delta(\lambda),\Delta(\mu)/N)$ non-zero for some submodule $N\subset\Delta(\mu)$. Then there exists $S\in\mathsf{M}(\lambda)$ such that $\mu\leq_{\boldsymbol{\varepsilon}_S}\lambda$. □

Proof. Since ϕ^{λ} is non-zero there exists – by Corollary 3.5 – a generator $z\in\Delta(\lambda)$ such that $\mathfrak{R}(\{\lambda\})\centerdot z=\Delta(\lambda)$. Then there exists $a\in\mathfrak{R}(\{\lambda\})$ such that $f(a\centerdot z)=a\centerdot f(z)\neq 0$, i.e. there exist $U,U'\in\mathsf{M}(\mu)$ such that $r_a(U,U')\neq 0$.

This implies that there exist $S,T\in\mathsf{M}(\lambda)$ such that for all $V\in\mathsf{M}(\mu)$ the expansion of $\mathtt{C}^{\lambda}_{S,T}\mathtt{C}^{\mu}_{U,V}$, using (2.6.d), contains a non-zero summand in $\mathfrak{R}(\{\mu\})$. Thus, $\mu\leq_{\boldsymbol{\varepsilon}_S}\lambda$. ■

As can be seen in Lemma 3.8, it is possible to have morphism in both "directions", and obtain $\lambda\leq_{\boldsymbol{\varepsilon}}\mu\leq_{\boldsymbol{\varepsilon}'}\lambda$. But we might still have $\lambda\neq\mu$ in case $\boldsymbol{\varepsilon}\neq\boldsymbol{\varepsilon}'$. This is in contrast to the framework of usual cellular algebras.

Let us give an alternative formulation of Lemma 3.8.

Lemma 3.9. Let $\lambda, \mu \in \mathsf{X}$ and $S, T \in \mathsf{M}(\lambda)$ such that $\mathsf{c}_{S,T}^{\lambda} \centerdot \Delta(\mu) \neq 0$ for some basis element $\mathsf{c}_{S,T}^{\lambda}$. Then $\mu \leq_{\varepsilon_S} \lambda$. □

Proof. By assumption there exists $U, V \in \mathsf{M}(\mu)$ such that the expansion of $\mathsf{c}_{S,T}^{\lambda}\mathsf{c}_{U,V}^{\mu}$, using (2.6.d), contains a non-zero summand in $\mathfrak{R}(\{\mu\})$. Thus, $\mu \leq_{\varepsilon_S} \lambda$. ■

Despite the fact that hom-spaces between cell modules are not as useful as in the case of usual cellular algebras, the following is surprisingly still true.

Proposition 3.10. For $\lambda \in \mathsf{X}_0$ it holds $\mathrm{End}_{\mathfrak{R}}(\Delta(\lambda)) = \mathbb{K}$. ■

Proof. We prove the following claim, which immediately proves the proposition.

3.10.Claim. Let $\lambda \in \mathsf{X}_0$, $\mu \in \mathsf{X}$ and let $N \subset \Delta(\mu)$ be some submodule. Then any $f \in \mathrm{Hom}_{\mathfrak{R}}(\Delta(\lambda), \Delta(\mu)/N)$ is of the form $f(x) = rx + N$ for some $r \in \mathbb{K}$.

Proof of 3.10.Claim. By assumption we can choose $y, y' \in \Delta(\lambda)$ such that $\phi^{\lambda}(y, y') = 1$. (Recall that we work over a field.) Fix u such that $f(y') = u + N$ and set $r = \phi^{\lambda}(y, u)$. Then $f(x) = f(\phi^{\lambda}(y, y')x) = \Theta^{\lambda}(x \otimes y) \centerdot f(y') = \Theta^{\lambda}(x \otimes y) \centerdot u + N$. Hence, we get $f(x) = \phi^{\lambda}(y, u)x + N = rx + N$. ■

3C. **Projective modules.** We have already seen in Section 3B that some statements from usual cellular algebras are quite different in the relative setup. Even more, from now on the relative setup needs some very careful treatment of the involved partial orders, all of which is trivial in the usual setup.

We start with some statements about idempotents. In the following we call an idempotent $e \in \mathfrak{R}$ an *idempotent summand of* $\varepsilon \in \mathsf{E}$ if $\varepsilon e = e = e\varepsilon$. In this case we write $e \Subset \varepsilon$.

Remark 3.11. By Lemma 2.10, at least in case $|\mathsf{X}| < \infty$, we can restrict our attention to $e \Subset \varepsilon$: Since we get a(n orthogonal) decomposition of the unit, we can find $\varepsilon \in \mathsf{E}$ for all indecomposable projectives P of $\mathfrak{R}$ such that $P \cong \mathfrak{R}e$ for primitive $e \Subset \varepsilon$. Thus, up to isomorphism, it suffices to study the projectives of the form $\mathfrak{R}e$ for $e \Subset \varepsilon$. ▲

Lemma 3.12. Let $e \Subset \varepsilon$ and I_{ε} an $<_{\varepsilon}$-ideal. Then the following hold.

(3.12.a) $e\mathfrak{R}(\{\lambda\}) \subset \mathfrak{R}(\leq_{\varepsilon} \lambda) \supset \mathfrak{R}(\{\lambda\})e$.

(3.12.b) $e\mathfrak{R}(\mathrm{I}_{\varepsilon}) \subset \mathfrak{R}(\mathrm{I}_{\varepsilon}) \supset \mathfrak{R}(\mathrm{I}_{\varepsilon})e$.

(3.12.c) $\mathfrak{R}(\mathrm{I}_{\varepsilon})e = \mathfrak{R}(\mathrm{I}_{\varepsilon}) \cap \mathfrak{R}e$, and $e\mathfrak{R}(\mathrm{I}_{\varepsilon}) = \mathfrak{R}(\mathrm{I}_{\varepsilon}) \cap e\mathfrak{R}$.

(3.12.d) $e \in \mathbb{K}\{\mathsf{c}_{S,T}^{\lambda} \mid \lambda \in \mathsf{X}, S, T \in \mathsf{M}(\lambda), \varepsilon_S = \varepsilon_T = \varepsilon\}$. □

Proof. *(3.12.a).* By (2.1-c1) and (2.6.a), since $\varepsilon e = e = e\varepsilon$ implies that $e \in \varepsilon\mathfrak{R}\varepsilon$.

(3.12.b). This follows from (3.12.a) since I_{ε} is an $<_{\varepsilon}$-ideal.

(3.12.c). We only prove the first statement, the second is obtained by applying *. By definition we get $\mathfrak{R}(\mathrm{I}_{\varepsilon})e \subset \mathfrak{R}e$, and by (3.12.a) we get $\mathfrak{R}(\mathrm{I}_{\varepsilon})e \subset \mathfrak{R}(\mathrm{I}_{\varepsilon})$. Hence, the left-hand side is contained in the right-hand side. Let $ae \in \mathfrak{R}(\mathrm{I}_{\varepsilon}) \cap \mathfrak{R}e$. We expand and – by assumption – obtain $ae = \sum_{\mu \in \mathrm{I}_{\varepsilon}, S,T \in \mathsf{M}(\mu)} r_{\mu,S,T}\mathsf{c}_{S,T}^{\mu}$ for some scalers $r_{\mu,S,T} \in \mathbb{K}$. Thus,

$$ae = (ae)e = \textstyle\sum_{\mu \in \mathrm{I}_{\varepsilon}, S,T \in \mathsf{M}(\mu)} r_{\mu,S,T}\mathsf{c}_{S,T}^{\mu}e \in \mathfrak{R}(\mathrm{I}_{\varepsilon})e.$$

It follows that the right-hand side is also contained in the left-hand side.

(3.12.d). This follows immediately from Corollary 2.7 by assumption on e. ■

Definition 3.13. For $e \Subset \varepsilon$ we define a partial order $<_e$ on X as being $<_{\varepsilon}$. ▲

We write $<_e=<_{\varepsilon}$ etc. in the following.

If the partial order with respect to which an ideal in X is defined agrees with the partial order $<_e$ for some $e\Subset\varepsilon$, then we can define submodules inside the corresponding projective module $P_e=\mathfrak{R}e$ to obtain suitable filtrations.

Lemma 3.14. Let $e\Subset\varepsilon$ and I_{ε} a $<_{\varepsilon}$-ideal. Then $\mathfrak{R}(\mathrm{I}_{\varepsilon})e$ is a submodule.

In case $|\mathsf{X}|<\infty$, there exists a filtration $P_e=P_0\supset P_1\supset\dots\supset P_r=\{0\}$ such that $P_i/P_{i+1}=P_e(\{\lambda_i\})$ for some $\lambda_i\in\mathsf{X}$. □

Hereby, similarly to (2-1), we let $P_e(\{\lambda\})=\mathfrak{R}(\leq_e\lambda)e/\mathfrak{R}(<_e\lambda)e$.

Proof. For $\mathtt{C}^{\lambda}_{S,T}\in\mathfrak{R}(\mathrm{I}_{\varepsilon})$ we have

$$a\mathtt{C}^{\lambda}_{S,T}e=\textstyle\sum_{S'\in\mathsf{M}(\lambda)}r_a(S',S)\mathtt{C}^{\lambda}_{S',T}e+(\dagger)$$

with $(\dagger)\in\mathfrak{R}(<_{\varepsilon_T}\lambda)\varepsilon_Te$ by (2.1.d). Then either $\varepsilon_Te=0$ in case $\varepsilon_T\neq\varepsilon$, and the extra terms just vanish, or $<_e=<_{\varepsilon_T}$ and $\varepsilon_Te=e$. Hence, $(\dagger)\in\mathfrak{R}(\mathrm{I}_{\varepsilon})$.

Last, choose a maximal chain of $<_e$-ideals – which exists by $|\mathsf{X}|<\infty$ – and the statement about the filtration follows immediately. ■

Analogously to Lemma 2.14, we let $\Gamma^{\lambda}\colon\Delta(\lambda)\otimes(\Delta(\lambda)^{\star}\centerdot e)\to P_e(\{\lambda\})$ defined via $\Gamma^{\lambda}(\mathtt{M}^{\lambda}_S\otimes(\mathtt{M}^{\lambda}_T\centerdot e))=\mathtt{C}^{\lambda}_{S,T}e$. (Below we write $\mathtt{M}^{\lambda}_S\otimes\mathtt{M}^{\lambda}_T\centerdot e$ etc. for short.)

Note that we will show the fact that Γ^{λ} is well-defined.

Proposition 3.15. Let $\lambda\in\mathsf{X}$ and $e\Subset\varepsilon$. Then Γ^{λ} is an $\mathfrak{R}$-module isomorphism. If additionally $\lambda\in\mathsf{X}_0$, then $\mathrm{Hom}_{\mathfrak{R}}(P_e(\{\lambda\}),\Delta(\lambda))\cong\mathrm{Hom}_{\mathbb{K}}(\Delta(\lambda)^{\star}\centerdot e,\mathbb{K})$. □

Proof. Well-definedness of Γ^{λ}. Define $\overline{\Gamma^{\lambda}}(\mathtt{M}^{\lambda}_S,\mathtt{M}^{\lambda}_T\centerdot e)=\mathtt{C}^{\lambda}_{S,T}e$ and extend bilinearly to obtain $\overline{\Gamma^{\lambda}}\colon\Delta(\lambda)\times\Delta(\lambda)^{\star}\centerdot e\to P_e(\{\lambda\})$. If $\overline{\Gamma^{\lambda}}$ is well-defined, then it is by definition bilinear. So let $\sum_{T\in\mathsf{M}(\lambda)}r_T(\mathtt{M}^{\lambda}_S,\mathtt{M}^{\lambda}_T\centerdot e)=0$ for some scalar $r_T\in\mathbb{K}$ and some element $[\mathtt{M}^{\lambda}_S,\mathtt{M}^{\lambda}_T\centerdot e]\in\Delta(\lambda)\times\Delta(\lambda)^{\star}\centerdot e$. Then

$$\overline{\Gamma^{\lambda}}\left(\textstyle\sum_{T\in\mathsf{M}(\lambda)}r_T[\mathtt{M}^{\lambda}_S,\mathtt{M}^{\lambda}_T\centerdot e]\right)=\textstyle\sum_{T,T'\in\mathsf{M}(\lambda)}r_Tr_{e^{\star}}(T',T)[\mathtt{M}^{\lambda}_S,\mathtt{M}^{\lambda}_{T'}].$$

Hence, $\sum_{T\in\mathsf{M}(\lambda)}r_Tr_{e^{\star}}(T',T)=0$ for all $T'\in\mathsf{M}(\lambda)$, and we have

$$\begin{aligned}\overline{\Gamma^{\lambda}}\left(\textstyle\sum_{T\in\mathsf{M}(\lambda)}r_T[\mathtt{M}^{\lambda}_S,\mathtt{M}^{\lambda}_T\centerdot e]\right)&\textstyle\sum_{T\in\mathsf{M}(\lambda)}\mathtt{C}^{\lambda}_{S,T}e\\&=\textstyle\sum_{T,T'\in\mathsf{M}}r_Tr_{e^{\star}}(T',T)\mathtt{C}^{\lambda}_{S,T'}+(\dagger).\end{aligned}$$

Hereby $(\dagger)\in\mathfrak{R}(<_{\varepsilon_S}\lambda)$ by (2-right) and $(\dagger)\in\mathfrak{R}(\leq_e\lambda)$ by (3.12.b), together giving $(\dagger)\in\mathfrak{R}(<_e\lambda)$. Since we also have that $(\dagger)=(\dagger)e$, it follows that $(\dagger)\in\mathfrak{R}e$. By (3.12.c) we then get that $(\dagger)\in\mathfrak{R}(<_e\lambda)e$ and so it vanishes in $P_e(\lambda)$. Thus, $\overline{\Gamma^{\lambda}}$ is well-defined and consequently Γ^{λ} as well.

Surjectivity of Γ^{λ}. This is immediate by noting that – due to (3.12.c) – $P_e(\{\lambda\})$ is generated by elements of the form $\mathtt{C}^{\lambda}_{S,T}e$ for $S,T\in\mathsf{M}(\lambda)$ and these are in the image of Γ^{λ}.

Injectivity of Γ^{λ}. Let $\sum_{S,T\in\mathsf{M}}r_{S,T}\mathtt{M}^{\lambda}_S\otimes\mathtt{M}^{\lambda}_T\centerdot e$ be in the kernel of Γ^{λ} for some scalars $r_{S,T}\in\mathbb{K}$, i.e. $\sum_{S,T\in\mathsf{M}(\lambda)}r_{S,T}\mathtt{C}^{\lambda}_{S,T}e\in\mathfrak{R}(<_e\lambda)e$. By (3.12.c) we have $\mathfrak{R}(<_e\lambda)e=\mathfrak{R}(<_e\lambda)\cap\mathfrak{R}e$ and so expanding with (2-right) we obtain

$$\textstyle\sum_{S,T\in\mathsf{M}(\lambda)}r_{S,T}\mathtt{C}^{\lambda}_{S,T}e=\sum_{S,T,T'\in\mathsf{M}(\lambda)}r_{S,T}r_{e^{\star}}(T',T)\mathtt{C}^{\lambda}_{S,T'}+(\dagger),$$

with $(\dagger) \in \mathfrak{R}(<_e \lambda)$ by (2-right) and (3.12.b). Thus, $\sum_{S,T} r_{S,T} r_{e^\star}(T',T) = 0$ for all $T' \in \mathsf{M}(\lambda)$, due to (3.12.c). This in turn implies that

$$\textstyle\sum_{S,T\in\mathsf{M}(\lambda)} r_{S,T}\mathtt{M}_S^\lambda \otimes \mathtt{M}_T^\lambda \centerdot e = \sum_{S,T,T'\in\mathsf{M}(\lambda)} r_{S,T} r_{e^\star}(T',T)\mathtt{M}_S^\lambda \otimes \mathtt{M}_{T'}^\lambda = 0.$$

Hence, Γ^λ is injective.

Γ^λ *is a* $\mathfrak{R}$*-module map.* For Γ^λ to be a $\mathfrak{R}$-module map we observe that

$$a\mathtt{C}_{S,T}^\lambda e = \textstyle\sum_{S'\in\mathsf{M}(\lambda)} r_a(S',S)\mathtt{C}_{S',T}^\lambda e + (\dagger)e,$$

where $(\dagger)e \in \mathfrak{R}(<_{\varepsilon_S} \lambda)\varepsilon_S e \subset \mathfrak{R}(<_e \lambda)e$, which this is zero in $P_e(\{\lambda\})$. Thus, Γ^λ is a $\mathfrak{R}$-module map.

Finally, for the isomorphism, let $\lambda \in \mathsf{X}_0$. By the above

$$\begin{aligned}\mathrm{Hom}_{\mathfrak{R}}(P_e(\{\lambda\}), \Delta(\lambda)) &\cong \mathrm{Hom}_{\mathfrak{R}}(\Delta(\lambda) \otimes \Delta(\lambda)^\star \centerdot e, \Delta(\lambda)) \\ &\cong \mathrm{Hom}_{\mathbb{K}}(\Delta(\lambda)^\star \centerdot e, \mathrm{End}_{\mathfrak{R}}(\Delta(\lambda))) \\ &\cong \mathrm{Hom}_{\mathbb{K}}(\Delta(\lambda)^\star \centerdot e, \mathbb{K}),\end{aligned}$$

where the second isomorphism is the tensor-hom adjunction, and the last isomorphism follows from Proposition 3.10. ■

In addition to statements about $P_e(\{\lambda\})$, we will also need some knowledge about slightly more general quotients of $\mathfrak{R}(\mathrm{I}_\varepsilon)e$.

Lemma 3.16. Let $e \Subset \varepsilon$ and I_ε an $<_\varepsilon$-ideal. Assume that I_ε contains $<_\varepsilon$-maximal elements $\lambda_1, \cdots, \lambda_r$ and let $\mathrm{I}'_\varepsilon = \mathrm{I}_\varepsilon \setminus \{\lambda_1, \cdots, \lambda_r\}$. Then

$$\mathfrak{R}(\mathrm{I}_\varepsilon)e/\mathfrak{R}(\mathrm{I}'_\varepsilon)e \cong P_e(\{\lambda_1\}) \oplus \cdots \oplus P_e(\{\lambda_r\}),$$

which is an isomorphism of $\mathfrak{R}$-modules. □

Proof. Let $\mathrm{I}_\varepsilon^{\leq_e \lambda_k} = \{\mu \in \mathrm{I}_\varepsilon \mid \mu \leq_e \lambda_k\}$ for $k = 1, \cdots, r$, and define $\mathrm{I}_\varepsilon^{<_e \lambda_k}$ analogously. By assumption, we have $\mathfrak{R}(\mathrm{I}_\varepsilon) = \sum_{k=1}^r \mathfrak{R}(\mathrm{I}_\varepsilon^{\leq_e \lambda_k})$ and $\mathfrak{R}(\mathrm{I}'_\varepsilon) = \sum_{k=1}^r \mathfrak{R}(\mathrm{I}_\varepsilon^{<_e \lambda_k})$. Additionally, we clearly have $\mathfrak{R}(\mathrm{I}'_\varepsilon) \cap \mathfrak{R}(\mathrm{I}_\varepsilon^{\leq_e \lambda_k}) = \mathfrak{R}(\mathrm{I}_\varepsilon^{<_e \lambda_k})$. Thus – using (3.12.c) – we obtain $\mathfrak{R}(\mathrm{I}'_\varepsilon)e \cap \mathfrak{R}(\mathrm{I}_\varepsilon^{\leq_e \lambda_k})e = \mathfrak{R}(\mathrm{I}_\varepsilon^{<_e \lambda_k})e$. Hence, the image of $\mathfrak{R}(\mathrm{I}_\varepsilon^{\leq_e \lambda_k})e$ in $\mathfrak{R}(\mathrm{I}_\varepsilon)e/\mathfrak{R}(\mathrm{I}'_\varepsilon)e$ is isomorphic to $P_e(\{\lambda_k\}) = \mathfrak{R}(\mathrm{I}_\varepsilon^{\leq_e \lambda_k})e/\mathfrak{R}(\mathrm{I}_\varepsilon^{<_e \lambda_k})e$.

In addition, for $1 \leq k, l \leq r$ and $k \neq l$,

$$\mathfrak{R}(\mathrm{I}_\varepsilon^{\leq_e \lambda_k})e \cap \mathfrak{R}(\mathrm{I}_\varepsilon^{\leq_e \lambda_l})e = \mathfrak{R}(\mathrm{I}_\varepsilon^{\leq_e \lambda_k}) \cap \mathfrak{R}(\mathrm{I}_\varepsilon^{\leq_e \lambda_l}) \cap \mathfrak{R}e = \mathfrak{R}(\mathrm{I}_\varepsilon^{<_e \lambda_k})e \cap \mathfrak{R}(\mathrm{I}_\varepsilon^{<_e \lambda_l})e.$$

Thus, the images of $\mathfrak{R}(\mathrm{I}_\varepsilon^{\leq_e \lambda_k})e$ and $\mathfrak{R}(\mathrm{I}_\varepsilon^{\leq_e \lambda_l})e$ in $\mathfrak{R}(\mathrm{I}_\varepsilon)e/\mathfrak{R}(\mathrm{I}'_\varepsilon)e$ have trivial intersection. Together this gives the statement. ■

3D. **Classification of simples.** Altogether we are now ready to prove the main statement of this section.

Theorem 3.17. Let $|\mathsf{X}| < \infty$. The set $\{[L(\lambda)] \mid \lambda \in \mathsf{X}_0\}$ gives a complete, non-redundant set of isomorphism classes of simple $\mathfrak{R}$-modules. □

Proof. There are three statements to be proven: That the $L(\lambda)$'s are simple, that all simples appear and that $L(\lambda) \cong L(\mu)$ if and only if $\lambda = \mu$.

Simplicity. By Proposition 3.6, the $L(\lambda)$ are simple $\mathfrak{R}$-modules.

Completeness. Let $e\Subset\boldsymbol{\varepsilon}$, with e being primitive. Then the head of its associated indecomposable projective P_e is simple, and we can obtain every simple module by considering the heads of the indecomposable projectives of $\mathfrak{R}$.

Let I_P denote the $<_e$-ideal in X generated by $\{\lambda\in\mathsf{X}\mid P_e(\{\lambda\})\neq 0\}$. Thus, $P_e=\mathfrak{R}(\mathrm{I}_P)e$ and – by (3.12.c) and by applying * – one has $e,e^\star\in\mathfrak{R}(\mathrm{I}_P)$.

Let $\lambda_{\max}\in\mathrm{I}_P$ be $<_e$-maximal. Then – by construction – $P_e(\{\lambda_{\max}\})\neq 0$.

3.17.Claim.a. The form $\phi^{\lambda_{\max}}$ is non-zero, i.e. $\lambda_{\max}\in\mathsf{X}_0$.

Proof of 3.17.Claim.a. Assume $\phi^{\lambda_{\max}}$ to be zero. By Lemma 3.4 we know that

$$\mathtt{C}^{\lambda_{\max}}_{U,V}\centerdot\mathtt{M}^{\lambda_{\max}}_T=\phi^{\lambda_{\max}}(\mathtt{M}^{\lambda_{\max}}_V,\mathtt{M}^{\lambda_{\max}}_T)\mathtt{M}^{\lambda_{\max}}_U=0,$$

for all $T,U,V\in\mathsf{M}(\lambda_{\max})$.

Expanding $e^\star=\sum_{\mu\in\mathrm{I}_P,S,T\in\mathsf{M}(\mu)}r(\mu,S,T)\mathtt{C}^{\mu}_{S,T}$ with $r(\mu,S,T)\in\mathbb{K}$, we see

$$\begin{aligned}e^\star\mathtt{C}^{\lambda_{\max}}_{V,U}&=\textstyle\sum_{\mu\in\mathrm{I}_P\setminus\lambda_{\max},S,T\in\mathsf{M}(\mu)}r(\mu,S,T)\mathtt{C}^{\mu}_{S,T}\mathtt{C}^{\lambda_{\max}}_{V,U}\\&=\textstyle\sum_{\mu\in\mathrm{I}\setminus\lambda_{\max},S,T\in\mathsf{M}(\mu)}\sum_{T'\in\mathsf{M}(\mu)}r(\mu,S,T)r_{\mathtt{C}^{\lambda_{\max}}_{V,U}}(T',T)\mathtt{C}^{\mu}_{S,T'}+(\dagger_\mu),\end{aligned}\tag{3-1}$$

where $(\dagger_\mu)\in\boldsymbol{\varepsilon}_U^\star\mathfrak{R}(<_{\boldsymbol{\varepsilon}_U}\mu)$ by (2-left). Hence, $e^\star(\dagger_\mu)\in e^\star\boldsymbol{\varepsilon}_U^\star\mathfrak{R}(<_{\boldsymbol{\varepsilon}_U}\mu)$. Recalling that $e\Subset\boldsymbol{\varepsilon}$, this is either zero if $\boldsymbol{\varepsilon}_U\neq\boldsymbol{\varepsilon}$, or $e^\star\boldsymbol{\varepsilon}_U^\star\mathfrak{R}(<_{\boldsymbol{\varepsilon}_U}\mu)=e^\star\mathfrak{R}(<_e\mu)\subset\mathfrak{R}(<_e\mu)$, with the final inclusion due to (3.12.a).

Multiplying the sum in (3-1) with $e^\star$ we obtain an element inside $e^\star\mathfrak{R}(\mathrm{I}_P\setminus\lambda_{\max})$, which is contained in $\mathfrak{R}(\mathrm{I}_P\setminus\lambda_{\max})$ by (3.12.a). Thus, $e^\star\mathtt{C}^{\lambda_{\max}}_{V,U}$ contains no summand in $\mathfrak{R}(\{\lambda_{\max}\})$ and we get $e^\star\centerdot\mathtt{M}^{\lambda_{\max}}_V=0$ for all $V\in\mathsf{M}(\lambda_{\max})$, implying $\Delta(\lambda_{\max})^\star\centerdot e=0$.

Since $P_e(\{\lambda_{\max}\})\cong\Delta(\lambda_{\max})\otimes\Delta(\lambda_{\max})^\star\centerdot e$ by Proposition 3.15, we thus obtain $P_e(\{\lambda_{\max}\})=0$, which is a contradiction to the choice of $\lambda_{\max}$ being a $<_e$-maximal element. Thus, $\phi^{\lambda_{\max}}$ is non-zero.

3.17.Claim.b. $\Delta(\lambda_{\max})$ is a quotient of $P_e(\{\lambda_{\max}\})$.

Proof of 3.17.Claim.b. First, 3.17.Claim.a and Proposition 3.15 imply that

$$\mathrm{Hom}_{\mathfrak{R}}(P_e(\{\lambda_{\max}\}),\Delta(\lambda_{\max}))\cong\mathrm{Hom}_{\mathbb{K}}(\Delta(\lambda_{\max})^\star\centerdot e,\mathbb{K})\neq 0.$$

Using this identification, choose a linear form f on $\Delta(\lambda_{\max})^\star\centerdot e$ and elements $xe\in\Delta(\lambda_{\max})^\star\centerdot e$ such that $f(xe)=1$ (recalling that we work over a field). Let now $z\in\Delta(\lambda_{\max})$ be a generator – which exists due to Lemma 3.4. Then, using again $P_e(\{\lambda_{\max}\})\cong\Delta(\lambda_{\max})\otimes\Delta(\lambda_{\max})^\star\centerdot e$, we obtain that f corresponds to the map sending $z\otimes xe$ to $f(xe)z=z$. Hence, $\Delta(\lambda_{\max})$ is a quotient of $P_e(\{\lambda_{\max}\})$.

By 3.17.Claim.b and Proposition 3.6, we get that $L(\lambda_{\max})$ is a quotient of $P_e(\{\lambda_{\max}\})$. With the choice of $\lambda_{\max}$ being $<_e$-maximal we have that $P_e(\{\lambda_{\max}\})$ is a quotient of P_e itself, and thus the head of P_e contains $L(\lambda_{\max})$. Since P_e is indecomposable, it has a simple head, which thus has to be $L(\lambda_{\max})$. So the completeness will follow after we have established 3.17.Claim.c:

3.17.Claim.c. There are no primitive idempotents e with $aea^{-1}\not\Subset\boldsymbol{\varepsilon}$ for all $\boldsymbol{\varepsilon}\in\mathsf{E}$ and all units $a\in\mathfrak{R}$.

Proof of 3.17.Claim.c. This follows from Lemma 2.10, see also Remark 3.11.

Non-redundancy. We continue to use the notation from above.

3.17.Claim.d. The ideal I_P has a unique $<_e$-maximal element.

Proof of 3.17.Claim.d. Assume that I_P has $<_e$-maximal elements $\lambda_0, \dots, \lambda_r$. Then for each of these we know that $P_e(\{\lambda_k\}) \neq 0$ and ϕ^{λ_k} is non-zero, i.e. $\Delta(\lambda_k)$ has a simple quotient. (This is 3.17.Claim.a.) Then – by Lemma 3.16 – we have that

$$\mathfrak{R}(\mathrm{I}_P)e/\mathfrak{R}(\mathrm{I}_P \setminus \{\lambda_0, \cdots, \lambda_r\})e \cong P_e(\{\lambda_0\}) \oplus \cdots \oplus P_e(\{\lambda_r\}).$$

This in turn implies that P_e has $L(\lambda_0) \oplus \cdots \oplus L(\lambda_r)$ as a quotient, which is a contradiction to P_e being indecomposable. Hence, the ideal I_P has a unique maximal element which we denote by $\lambda_{\max}$.

Now, 3.17.Claim.e will establish non-redundancy, which will finish the proof.

3.17.Claim.e. $L(\lambda) \cong L(\mu)$ implies $\lambda = \mu$ for $\lambda, \mu \in \mathsf{X}_0$.

Proof of 3.17.Claim.e. Without loss of generality, assume that λ is a $<_e$-maximal element in an ideal I_P for some indecomposable projective P_e corresponding to $e \Subset \boldsymbol{\varepsilon}$ primitive for some $\boldsymbol{\varepsilon} \in \mathsf{E}$. (This is sufficient since we already proved above that simples obtained for these elements of X give a complete set of isomorphism classes of simples.)

We first observe that we have a quotient map

$$\pi_\lambda \colon P_e \twoheadrightarrow L(\lambda) \xrightarrow{\cong} L(\mu),$$

with $z_\lambda = \pi_\lambda(e)$ being a generator of $L(\mu)$. Thus, one has $e \centerdot z_\lambda = z_\lambda$. Note now that $e \in \mathfrak{R}(\leq_e \lambda)$ since λ is unique $<_e$-maximal by 3.17.Claim.d. Thus, (3.12.d) implies that there exists $\eta \leq_e \lambda$, $S, T \in \mathsf{M}(\eta)$ with $\boldsymbol{\varepsilon}_S = \boldsymbol{\varepsilon}_T = \boldsymbol{\varepsilon}$ and $U, V \in \mathsf{M}(\mu)$ such that the product

$$\mathsf{c}^{\eta}_{S,T}\mathsf{c}^{\mu}_{U,V} \in \textstyle\sum_{T' \in \mathsf{M}(\eta)} r_{\mathsf{c}^{\mu}_{V,U}}(T', T)\mathsf{c}^{\eta}_{S,T'} + \boldsymbol{\varepsilon}^{\star}_S \mathfrak{R}(<_{\boldsymbol{\varepsilon}_S} \eta),$$

expanded using (2-right), contains a summand in $\mathfrak{R}(\{\mu\})$. Hence, with $\boldsymbol{\varepsilon}_S = \boldsymbol{\varepsilon}$ (giving $<_e = <_{\boldsymbol{\varepsilon}_S}$) it follows that $\mu \leq_e \eta \leq_e \lambda$.

On the other hand – by Lemma 3.4 – we have $\Delta(\mu) = \mathfrak{R}(\{\mu\}) \centerdot z$ for some generator $z \in \Delta(\mu)$ which gives another quotient map

$$\psi_\lambda \colon \Delta(\mu) \twoheadrightarrow L(\mu) \xrightarrow{\cong} L(\lambda).$$

Fix now z_λ as above and choose $y \in \Delta(\mu)$ with $\psi_\lambda(y) = z_\lambda$. Then there exists $a \in \mathfrak{R}(\{\mu\})$ such that $y = a \centerdot z$, but

$$\psi_\lambda((\boldsymbol{\varepsilon} a) \centerdot z) = \boldsymbol{\varepsilon} \centerdot \psi_\lambda(a \centerdot z) = \boldsymbol{\varepsilon} \centerdot \psi_\lambda(y) = \boldsymbol{\varepsilon} \centerdot z_\lambda = e \centerdot z_\lambda = z_\lambda,$$

so we can assume that $\boldsymbol{\varepsilon} a = a$ and $a \centerdot \psi_\lambda(z) \neq 0$. So there exist $S, T \in \mathsf{M}(\lambda)$ and $U, V \in \mathsf{M}(\mu)$ with $\boldsymbol{\varepsilon}_U = \boldsymbol{\varepsilon}$ such that

$$\mathsf{c}^{\mu}_{U,V}\mathsf{c}^{\lambda}_{S,T} \in \textstyle\sum_{T' \in \mathsf{M}(\eta)} r_{\mathsf{c}^{\lambda}_{T,S}}(V', V)\mathsf{c}^{\mu}_{U,V'} + \boldsymbol{\varepsilon}^{\star}_U \mathfrak{R}(<_{\boldsymbol{\varepsilon}_U} \mu),$$

expanded using (2-right), contains a summand in $\mathfrak{R}(\{\lambda\})$. Thus, with $\boldsymbol{\varepsilon}_U = \boldsymbol{\varepsilon}$ (giving $<_e = <_{\boldsymbol{\varepsilon}_U}$) it follows that $\lambda \leq_e \mu$.

Hence, altogether we have $\lambda = \mu$. ■

Note that the primitive idempotent e such that $\mathfrak{R}e$ has $L(\lambda)$ as its head is not unique. But if we demand the choice of an idempotent summand of some $\boldsymbol{\varepsilon}_\lambda \in \mathsf{E}$, then $\boldsymbol{\varepsilon}_\lambda$ is unique. In particular, the associated partial order $<_{\boldsymbol{\varepsilon}_\lambda}$ is independent of the choice of e. Thus – having Theorem 3.17 – we can define:

Definition 3.18. Let $|\mathsf{X}| < \infty$ and $\lambda \in \mathsf{X}_0$. We denote by $P(\lambda)$ the indecomposable projective module corresponding to $L(\lambda)$. ▲

The partial order associated to $P(\lambda)$ is denoted by $<_\lambda$.

Proposition 3.19. Let $\lambda \in \mathsf{X}_0$. Then $P(\lambda)$ has a filtration by cell modules $\Delta(\mu)$ such that $\mu \leq_\lambda \lambda$. □

Proof. By the proof of Theorem 3.17 we know that $P(\lambda) = \mathfrak{R}(\leq_\lambda \lambda)e$ for some $e \Subset \boldsymbol{\varepsilon}_\lambda$ primitive. The statement follows by Lemma 3.14 and the description of the subquotients as direct sums of cell modules from Proposition 3.15. ■

Examples illustrating Proposition 3.19 are the ones in Section 2E.

3E. **Reciprocity laws.** Throughout the rest of the section assume $|\mathsf{X}| < \infty$. Let $\lambda \in \mathsf{X}_0$ and $\mu \in \mathsf{X}$.

We denote by $\mathrm{d}_{\mu,\lambda} = [\Delta(\mu) : L(\lambda)]$ the *Jordan–Hölder multiplicity of* $L(\lambda)$ *in* $\Delta(\mu)$ and by $\mathcal{D} = \mathcal{D}(\mathfrak{R}) = (\mathrm{d}_{\mu,\lambda})_{\mu\in\mathsf{X},\lambda\in\mathsf{X}_0}$ the *decomposition matrix of* $\mathfrak{R}$.

In contrast to [GL96, Proposition 3.6], the matrix $\mathcal{D}$ is not upper triangular, cf. Example 3.24. But we have the following relative version.

Proposition 3.20. Let $\lambda \in \mathsf{X}_0$ and $\mu \in \mathsf{X}$. Then $\mathrm{d}_{\mu,\lambda} = 0$ unless $\mu \leq_\lambda \lambda$. Furthermore, we have $\mathrm{d}_{\lambda,\lambda} = 1$. □

Proof. Assume that $\mathrm{d}_{\mu,\lambda} \neq 0$. Then there exists a non-zero map $f\colon \Delta(\lambda) \to \Delta(\mu)/N$ for some submodule $N \subset \Delta(\mu)$. Corresponding to $L(\lambda)$ there exists some $\boldsymbol{\varepsilon} \in \mathsf{E}$ and $e \Subset \boldsymbol{\varepsilon}$ such that e acts non-trivial on $L(\lambda)$. Hence, e acts also non-trivial on $\Delta(\lambda)$, and furthermore $e \in \mathfrak{R}(\leq_\lambda \lambda)$. Since f is an $\mathfrak{R}$-module map, e also acts non-trivial on $\Delta(\mu)/N$, and hence also non-trivial on $\Delta(\mu)$. Thus, there exists $\eta \leq_\lambda \lambda$, $S, T \in \mathsf{M}(\eta)$ with $\boldsymbol{\varepsilon}_S = \boldsymbol{\varepsilon}$ such that $\mathsf{c}^\eta_{S,T} \centerdot \Delta(\mu) \neq 0$. Thus – by Lemma 3.9 – we have that $\mu \leq_\lambda \eta \leq_\lambda \lambda$.

Assume now that $\lambda = \mu \in \mathsf{X}_0$. Let $f\colon \Delta(\lambda) \to \Delta(\lambda)/N$ for some submodule N be a non-zero map. Then we know – by 3.10.Claim – that the map is a non-zero $\mathbb{K}$-multiple of the identity of $\Delta(\lambda)$ composed with the natural quotient map. Thus, f is always surjective and only in case of $N = \mathrm{rad}(\lambda)$ is the image simple. This gives $\mathrm{d}_{\lambda,\lambda} = 1$. ■

Lemma 3.21. Let $\lambda \in \mathsf{X}_0$ and $e \Subset \boldsymbol{\varepsilon}_\lambda$ primitive. Then $P(\lambda) \cong \mathfrak{R}e$ if and only if $\mathrm{I}_\lambda = \{\mu \in \mathsf{X} \mid \mu \leq_\lambda \lambda\}$ is the smallest $<_\lambda$-ideal such that $e \in \mathfrak{R}(\mathrm{I}_\lambda)$. □

Proof. $\Rightarrow$. Assuming that $P(\lambda) \cong \mathfrak{R}e$, we know that I_λ is an $<_\lambda$-ideal such that $e \in \mathfrak{R}(\mathrm{I}_\lambda)$, see the proof of Theorem 3.17. Assume now that I is another $<_\lambda$-ideal such that $e \in \mathfrak{R}(\mathrm{I})$. If $\lambda \in \mathrm{I}$ we are done, since $\mathrm{I}_\lambda \subset \mathrm{I}$. So assume $\lambda \notin \mathrm{I}$ and denote by $\langle \mathrm{I} \cup \lambda \rangle$ the $<_\lambda$-ideal generated by I and λ. Then $P(\{\lambda\}) = \mathfrak{R}(\langle \mathrm{I} \cup \lambda \rangle)e/\mathfrak{R}(\langle \mathrm{I} \cup \lambda \rangle \setminus \lambda)e = 0$, since $P(\lambda) = \mathfrak{R}(\mathrm{I})e$. This is a contradiction to $L(\lambda)$ being the quotient of $P(\lambda)$. Thus, I_λ is the smallest $<_\lambda$-ideal with the desired property.

$\Leftarrow$. For I_λ being the smallest $<_\lambda$-ideal with $e \in \mathfrak{R}(\mathrm{I}_\lambda)$, let $\mu \in \mathsf{X}_0$ such that $\mathfrak{R}e = P(\mu)$. Then I_μ is the smallest $<_\lambda$-ideal containing e, and thus – by assumption – equal to I_λ. Hence – by Theorem 3.17 – $P(\mu)$ has simple quotient $L(\lambda)$, giving $\mu = \lambda$. ■

Since for a primitive idempotent summand of $\boldsymbol{\varepsilon}_\lambda$, the minimal $<_\lambda$-ideal I such that $e \in \mathfrak{R}(\mathrm{I})$ is equal the minimal $<_\lambda$-ideal such that $e^\star \in \mathfrak{R}(\mathrm{I})$, the following is immediate.

Corollary 3.22. Let $\lambda \in \mathsf{X}_0$. If $P(\lambda) \cong \mathfrak{R}e$ for $e \Subset \boldsymbol{\varepsilon}_\lambda$, then $P(\lambda) \cong \mathfrak{R}e^\star$. ■

For $\lambda, \mu \in \mathsf{X}_0$ we denote by $\mathrm{c}_{\lambda,\mu} = [P(\lambda) : L(\mu)]$ the *Jordan–Hölder multiplicity of* $L(\mu)$ *in* $P(\lambda)$, and by $\mathcal{C} = \mathcal{C}(\mathfrak{R}) = (\mathrm{c}_{\lambda,\mu})_{\lambda,\mu\in\mathsf{X}_0}$ the *Cartan matrix of* $\mathfrak{R}$. (By Theorem 3.17 this coincides with the definition we used in Section 2E.)

Theorem 3.23. Let $\lambda \in \mathsf{X}_0$, $\mu \in \mathsf{X}$ and $e \Subset \boldsymbol{\varepsilon}_\lambda$ primitive such that $P(\lambda) = \mathfrak{R}e$.

(3.23.a) The multiplicity $\mathrm{d}_{\mu,\lambda}$ is equal to $\dim(\Delta(\mu)^{\star}.e)$.

(3.23.b) For $\mu \in \mathsf{X}_0$ it holds that

$$[P(\lambda) : L(\mu)] = \textstyle\sum_{\nu\in\mathsf{X},\nu\leq_{\lambda}\lambda,\nu\leq_{\mu}\mu}[\Delta(\nu) : L(\lambda)][\Delta(\nu) : L(\mu)].$$

(Or $\mathcal{C} = \mathcal{D}^{\mathrm{T}}\mathcal{D}$ written as matrices.) □

Proof. *(3.23.a)*. This is straightforward, since

$$\begin{aligned}\mathrm{d}_{\mu,\lambda} &= \dim(\mathrm{Hom}_{\mathfrak{R}}(P(\lambda), \Delta(\mu))) = \dim(\mathrm{Hom}_{\mathfrak{R}}(\mathfrak{R}e^{\star}, \Delta(\mu))) \\ &= \dim(e^{\star}\centerdot\Delta(\mu)) = \dim\Delta(\mu)^{\star}\centerdot e),\end{aligned}$$

with the second equality due to Corollary 3.22.

(3.23.b). Choose a maximal $<_{\lambda}$-ideal chain inside I_{λ}. Then we know for each subquotient $P(\{\nu\}) \cong \Delta(\nu) \otimes \Delta(\nu)^{\star}\centerdot e$ as left $\mathfrak{R}$-modules. Thus,

$$\mathrm{c}_{\lambda,\mu} = \textstyle\sum_{\nu\in\mathsf{X},\nu\leq_{\lambda}\lambda}\dim(\Delta(\nu)^{\star}\centerdot e)\mathrm{d}_{\nu,\mu} = \sum_{\nu\in\mathsf{X},\nu\leq_{\lambda}\lambda}\mathrm{d}_{\nu,\lambda}\mathrm{d}_{\nu,\mu},$$

where – by Proposition 3.20 – any summand is zero unless $\nu \leq_{\mu} \mu$ as well. ■

Example 3.24. Coming back to the examples from Section 2E, we have for $n = 3$

$$\begin{aligned}\mathcal{C}(\mathfrak{U}(\mathrm{A}_3)) &= \begin{pmatrix} 2&1&0\\1&2&1\\0&1&2\end{pmatrix} = \begin{pmatrix}1&1&0&0\\0&1&1&0\\0&0&1&1\end{pmatrix}\begin{pmatrix}1&0&0\\1&1&0\\0&1&1\\0&0&1\end{pmatrix} = \mathcal{D}(\mathfrak{U}(\mathrm{A}_3))^{\mathrm{T}}\mathcal{D}(\mathfrak{U}(\mathrm{A}_3)),\\ \mathcal{C}(\mathfrak{R}(\tilde{\mathrm{A}}_3)) &= \begin{pmatrix}2&1&1\\1&2&1\\1&1&2\end{pmatrix} = \begin{pmatrix}1&1&0\\0&1&1\\1&0&1\end{pmatrix}\begin{pmatrix}1&0&1\\1&1&0\\0&1&1\end{pmatrix} = \mathcal{D}(\mathfrak{R}(\tilde{\mathrm{A}}_3))^{\mathrm{T}}\mathcal{D}(\mathfrak{R}(\tilde{\mathrm{A}}_3)),\\ \mathcal{C}(\mathfrak{R}'(\tilde{\mathrm{A}}_3)) &= \begin{pmatrix}3&3&3\\3&3&3\\3&3&3\end{pmatrix} = \begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix} = \mathcal{D}(\mathfrak{R}'(\tilde{\mathrm{A}}_3))^{\mathrm{T}}\mathcal{D}(\mathfrak{R}'(\tilde{\mathrm{A}}_3)),\end{aligned}$$

(up to base change) and analogously for general n. Note that the decomposition matrices have an upper triangular shape for $\mathfrak{U}(\mathrm{A}_n)$, which is an usual cellular algebra. ▲

As a direct corollary of (3.23.b), we get a very easy to check, but weak, necessary criterion for an algebra to be relative cellular.

Corollary 3.25. If $\mathfrak{R}$ is relative cellular, then $\mathcal{C}$ is positive semidefinite. ■

As already discussed in detail in Section 2E, this is in contrast to the usual setup where $\mathcal{C}$ is positive definite, cf. Remark 2.18.

3F. **Further consequences.** For the next proposition, we denote by $\mathcal{D}$ the (*-twisted) duality on $\mathfrak{R}$-modules defined by $\mathcal{D}(M) = \mathrm{Hom}_{\mathbb{K}}(M^{\star}, \mathbb{K})$. Note that $\Delta(\lambda)$ is in general not isomorphic to $\mathcal{D}(\Delta(\lambda))$ as an $\mathfrak{R}$-module. But we have the following.

Proposition 3.26. Let $\lambda, \mu \in \mathsf{X}_0$. Then $\mathcal{D}L(\lambda) \cong L(\lambda)$ as $\mathfrak{R}$-modules. Further, $\mathrm{Ext}^i_{\mathfrak{R}}(L(\lambda), L(\mu)) \cong \mathrm{Ext}^i_{\mathfrak{R}}(L(\mu), L(\lambda))$ for all $i \in \mathbb{Z}_{\geq 0}$. □

Proof. Let $e \Subset \varepsilon_{\lambda}$ primitive such that $P(\lambda) = \mathfrak{R}e \cong \mathfrak{R}e^{\star}$. We claim that $P(\lambda)$ is a projective cover of the simple $\mathcal{D}L(\lambda)$. For $ae^{\star} \in \mathfrak{R}e^{\star}$ we define $\theta_{ae^{\star}}$ by $\theta_{ae^{\star}}(x) = x\centerdot(ae^{\star})$ for $x \in L(\lambda)^{\star}$. Here $x\centerdot(ae^{\star}) = ea^{\star}\centerdot x$ is an element in $eL(\lambda)$ which can be canonically identified with the endomorphism ring of $L(\lambda)$ which – by Proposition 3.10 – is $\mathbb{K}$. Thus, $\theta_{ae^{\star}}$ defines a linear form on $L(\lambda)^{\star}$. Clearly, the map $ae^{\star} \mapsto \theta_{ae^{\star}}$ is not the zero map, hence it is surjective and so $P(\lambda)$ is the projective cover of $\mathcal{D}L(\lambda)$.

Using $\mathrm{Ext}^i_{\mathfrak{R}}(L(\lambda), L(\mu)) \cong \mathrm{Ext}^i_{\mathrm{mod}-\mathfrak{R}}(L(\lambda)^{\star}, L(\mu)^{\star})$, the latter being in right $\mathfrak{R}$-modules, we obtain the statement about Ext-groups since vector space duality gives a contravariant equivalence between left and right modules for a finite-dimensional algebras. ■

Remark 3.27. As a corollary of Proposition 3.26, the Ext-quiver of a relative cellular algebra has a symmetric form. This is a well-known fact in the usual setting. ▲

Last, the semisimplicity criterion for a relative cellular algebra is as in [GL96, Theorem 3.8], and the proof – by using the results from Section 3E – is identical (and omitted).

Proposition 3.28. Let $\mathfrak{R}$ be a relative cellular algebra. Then the following are equivalent.

(3.28.a) $\mathfrak{R}$ is semisimple.

(3.28.b) The cell modules $\Delta(\lambda)$ for $\lambda \in \mathsf{X}_0$ are simple.

(3.28.c) The subspace $\mathrm{rad}(\lambda) = 0$ for all $\lambda \in \mathsf{X}$. ■

Example 3.29. None of the algebras from Section 2E, nor $\mathfrak{A}rc_n^{\mathrm{ann}}$ for $n \in \mathbb{Z}_{>0}$ (which we meet in Section 5) are semisimple. There are various ways to see this, but using Proposition 3.28 this follows since the simples are all of dimension one, while the cell modules are not. ▲

4. An extended example I: The restricted enveloping algebra of $\mathfrak{sl}_2$

Throughout this section let $\mathbb{K}$ be any field of positive characteristic p.

4A. **The algebra.** We let $\mathbb{F}_p$ be the prime field of $\mathbb{K}$, and we also use the set $\mathsf{F}_p = \{0, 1, \ldots, p-2, p-1\} \subset \mathbb{Z}_{\geq 0}$ underlying $\mathbb{F}_p$. (Using the identification $\mathbb{F}_p = \mathsf{F}_p$, we will sometimes read modulo p.)

Definition 4.1. The *restricted enveloping algebra of* $\mathfrak{sl}_2$, denoted by $\mathfrak{u}_0(\mathfrak{sl}_2)$, is the associative, unital algebra generated by $\mathcal{E}, \mathcal{F}, \mathcal{H}$ subject to

$$\mathcal{H}\mathcal{E} - \mathcal{E}\mathcal{H} = 2\mathcal{E}, \quad \mathcal{H}\mathcal{F} - \mathcal{F}\mathcal{H} = -2\mathcal{F}, \quad \mathcal{E}\mathcal{F} - \mathcal{F}\mathcal{E} = \mathcal{H}, \tag{4-1}$$

$$\mathcal{E}^p = \mathcal{F}^p = \mathcal{H}^p - \mathcal{H} = 0. \tag{4-p$\mathfrak{sl}_2$}$$

Said otherwise, $\mathfrak{u}_0(\mathfrak{sl}_2)$ is the usual enveloping algebra of $\mathfrak{sl}_2$ modulo (4-p$\mathfrak{sl}_2$). ▲

Remark 4.2. Our main source for the basics about $\mathfrak{u}_0(\mathfrak{sl}_2)$ are [FP88] and [Jan04]. (E.g., Definition 4.1 are taken from therein.) Note that $\mathfrak{u}_\chi(\mathfrak{sl}_2)$ can be defined for a choice of $\chi \in \mathfrak{sl}_2^*$. But, as we will see below, it is crucial for us that $\chi = 0$. ▲

Recall the following *PBW theorem*, cf. [FP88, Section 1] or [Jan04, Section A.3]:

Theorem 4.3. The set

$$\{\mathcal{F}^x \mathcal{H}^y \mathcal{E}^z \mid x, y, z \in \mathsf{F}_p\} \tag{4-2}$$

is a basis of $\mathfrak{u}_0(\mathfrak{sl}_2)$. ■

Our relative cellular basis for $\mathfrak{u}_0(\mathfrak{sl}_2)$ will be an idempotent version of (4-2). For this we need the following weight idempotents. Let $\lambda \in \mathsf{F}_p$ and define

$$1_\lambda = -\textstyle\prod_{\mu \in \mathbb{F}_p, \mu \neq \lambda} (\mathcal{H} - \mu).$$

Lemma 4.4. $\{1_\lambda \mid \lambda \in \mathsf{F}_p\}$ is set of pairwise orthogonal idempotents summing to 1. □

We stress that the 1_λ's are not primitive idempotents of $\mathfrak{u}_0(\mathfrak{sl}_2)$, but rather the primitive idempotents of the semisimple subalgebra spanned by the $\mathcal{H}$'s.

Proof. Observe that $\mathit{1}_\lambda$ is a degree $p-1$ polynomial in $\mathcal{H}$ and therefore determined by its values in $\mathbb{F}_p$. Now, substituting $\mathcal{H}$ with any element of $\mathbb{F}_p$, we see – by Wilson's theorem – that $\mathit{1}_\lambda$ is an idempotent. Similarly, orthogonality follows from Fermat's little theorem. Last – by construction – $\sum_{\lambda\in\mathsf{X}} \mathit{1}_\lambda$ evaluates for any substitution $\mathcal{H} \mapsto \mu \in \mathbb{F}_p$ to 1. ■

The following tedious calculations, which we will use throughout, are omitted.

Lemma 4.5. Let $\lambda \in \mathsf{F}_p$ and $S, T \in \mathsf{F}_p$.

(4.5.a) For $k \in \mathsf{F}_p$ we have

$$\mathcal{H}^k\mathcal{E}^T = \mathcal{E}^T(\mathcal{H}+2T)^k, \quad \mathcal{H}^k\mathcal{F}^S = \mathcal{F}^S(\mathcal{H}-2S)^k,$$
$$\mathit{1}_\lambda\mathcal{E} = \mathcal{E}\mathit{1}_{\lambda-2}, \quad \mathcal{E}\mathit{1}_\lambda = \mathit{1}_{\lambda+2}\mathcal{E}, \quad \mathit{1}_\lambda\mathcal{F} = \mathcal{F}\mathit{1}_{\lambda+2}, \quad \mathcal{F}\mathit{1}_\lambda = \mathit{1}_{\lambda-2}\mathcal{F}, \quad \mathcal{H}\mathit{1}_\lambda = \lambda\mathit{1}_\lambda = \mathit{1}_\lambda\mathcal{H}.$$

(4.5.b) We have

$$\mathcal{E}^T\mathcal{F}^S\mathit{1}_\lambda = \textstyle\sum_{j=0}^{\min(S,T)} \frac{S!T!}{(S-j)!(T-j)!}\binom{T-S+\lambda}{j}\mathcal{F}^{S-j}\mathcal{E}^{T-j}\mathit{1}_\lambda,$$
$$\mathcal{F}^S\mathcal{E}^T\mathit{1}_\lambda = \textstyle\sum_{j=0}^{\min(S,T)} \frac{S!T!}{(S-j)!(T-j)!}\binom{S-T-\lambda}{j}\mathcal{E}^{T-j}\mathcal{F}^{S-j}\mathit{1}_\lambda,$$

with usual factorials and binomials taken modulo p. ■

Remark 4.6. For $p = 2$ it is – by Lemma 4.5 – not hard to see that $\mathfrak{u}_0(\mathfrak{sl}_2)$ is isomorphic to a direct sum of $\mathbb{K}[X,Y]/(X^2,Y^2)$ and a semisimple algebra. Thus, $\mathfrak{u}_0(\mathfrak{sl}_2)$ is already cellular in the usual sense, and we from now on assume that $p > 2$. ▲

4B. **The cell datum.** Next, we want to define the relative cell datum for $\mathfrak{u}_0(\mathfrak{sl}_2)$. To this end, we let $\mathsf{X} = \mathsf{F}_p$ and $\mathsf{M}(\lambda) = \mathsf{F}_p$ for all $\lambda \in \mathsf{X}$. Moreover – by Lemma 4.4 – we can let $\mathsf{E} = \{\mathit{1}_\lambda \mid \lambda \in \mathsf{X}\}$ be our idempotent set.

Further, we let $\mathsf{C}^\lambda_{S,T} = \mathcal{F}^S\mathit{1}_\lambda\mathcal{E}^T$, and set $(\mathcal{F}^S\mathit{1}_\lambda\mathcal{E}^T)^\star = \mathcal{F}^T\mathit{1}_\lambda\mathcal{E}^S$. And finally, let the partial orders $\mathsf{O} = \{<_{\mathit{1}_\lambda} \mid \lambda \in \mathsf{X}\}$, on X, be defined via

$$\lambda + 2(p-1) <_{\mathit{1}_\lambda} \cdots <_{\mathit{1}_\lambda} \lambda + 4 <_{\mathit{1}_\lambda} \lambda + 2 <_{\mathit{1}_\lambda} \lambda,$$

and $\varepsilon_S = \mathit{1}_{\lambda+2S}$ for $S \in \mathsf{M}(\lambda)$. Note that these partial orders on X are well-defined since 2 generates $\mathbb{F}_p$ since we assume that $p > 2$.

To summarize, we have our cell datum

$$(\mathsf{X}, \mathsf{M}, \mathsf{C}, {}^\star, \mathsf{E}, \mathsf{O}, \epsilon). \tag{4-3}$$

A direct consequence of Lemma 4.5 is:

Lemma 4.7. Let $\mathsf{C}^\lambda_{S+1,T} = \mathsf{C}^\lambda_{S-1,T} = \mathsf{C}^\lambda_{S,T+1} = 0$ in case $S, T \notin \mathsf{F}_p$. Then

$$\mathcal{E}\,\mathsf{C}^\lambda_{S,T} = S(1-S+\lambda)\mathsf{C}^\lambda_{S-1,T} + \mathsf{C}^{\lambda+2}_{S,T+1}, \quad \mathcal{F}\,\mathsf{C}^\lambda_{S,T} = \mathsf{C}^\lambda_{S+1,T}, \quad \mathcal{H}\,\mathsf{C}^\lambda_{S,T} = (\lambda-2S)\mathsf{C}^\lambda_{S,T},$$

Similar formulas hold for the right action of $\mathfrak{u}_0(\mathfrak{sl}_2)$ on the $\mathsf{C}^\lambda_{S,T}$'s. ■

4C. **The cases $p = 3$ exemplified.**

Example 4.8. Let $p = 3$. Then $\mathit{1}_0 = -(\mathcal{H}-1)(\mathcal{H}-2)$, $\mathit{1}_1 = -(\mathcal{H}-0)(\mathcal{H}-2)$ and $\mathit{1}_2 = -(\mathcal{H}-0)(\mathcal{H}-1)$. Moreover, the partial orders are

$$\mathsf{X} = \{\boxed{1} <_{\mathit{1}_0} \boxed{2} <_{\mathit{1}_0} \boxed{0}\} = \{\boxed{2} <_{\mathit{1}_1} \boxed{0} <_{\mathit{1}_1} \boxed{1}\} = \{\boxed{0} <_{\mathit{1}_2} \boxed{1} <_{\mathit{1}_2} \boxed{2}\}.$$

Further, $\mathit{1}_\mu\mathfrak{u}_0(\mathfrak{sl}_2)\mathit{1}_\mu$ consists of elements $\mathcal{F}^S\mathit{1}_\lambda\mathcal{E}^S$ such that $\lambda = \mu - 2S$. Having all this, it is easy to see that (4-3) defines a cell datum for $\mathfrak{u}_0(\mathfrak{sl}_2)$.

We get projectives and cell modules (here exemplified in case $\lambda = 0$):

(4-4)
$$
\begin{array}{c|ccc|}
 & & \mathfrak{u}_0(\mathfrak{sl}_2)\, 1_0 & \\
\Delta(0) & \mathcal{F}^2 1_0\, 1_0 & \mathcal{F} 1_0\, 1_0 & 1_\lambda\, 1_0 \\
\Delta(2) & \mathcal{F}^2 1_2 \mathcal{E}\, 1_0 & \mathcal{F} 1_2 \mathcal{E}\, 1_0 & 1_2 \mathcal{E}\, 1_0 \\
\Delta(1) & \mathcal{F}^2 1_1 \mathcal{E}^2\, 1_0 & \mathcal{F} 1_1 \mathcal{E}^2\, 1_0 & 1_1 \mathcal{E}^2\, 1_0
\end{array}
\qquad
\Delta(1)\colon
\begin{tikzcd}
\mathtt{c}^1_{2,2} \arrow[loop above, "0"] \arrow[r, shift right, "0"'] & \mathtt{c}^1_{1,2} \arrow[loop above, "2"] \arrow[l, shift right, "1"'] \arrow[r, shift right, "1"'] & \mathtt{c}^1_{0,2} \arrow[loop above, "1"] \arrow[l, shift right, "1"']
\end{tikzcd}
\quad
\xrightarrow{\mathcal{E}}\ \xleftarrow{\mathcal{F}}\ \overset{\mathcal{H}}{\curvearrowright}
$$

all of which are nine respectively three dimensional. The Δ's are isomorphic to the so-called *baby Verma modules of highest weight* λ. For example, the cell module $\Delta(1)$ in $\mathfrak{u}_0(\mathfrak{sl}_2)1_0$ is the left $\mathfrak{u}_0(\mathfrak{sl}_2)$-module as displayed in (4-4).

In order to get the simples L, we calculate the radical and then we use Theorem 3.17. Note that, the pairing $\phi^\lambda(\mathcal{F}^S 1_\lambda, \mathcal{F}^T 1_\lambda)$ is zero unless $S = T$. For $S = T$ we get:

$$
\Delta(0)\colon \begin{cases} 1, & \text{if } S = T = 0, \\ 0, & \text{if } S = T = 1, \\ 0, & \text{if } S = T = 2, \end{cases} \quad \Delta(1)\colon \begin{cases} 1, & \text{if } S = T = 0, \\ 1, & \text{if } S = T = 1, \\ 0, & \text{if } S = T = 2, \end{cases} \quad \Delta(2)\colon \begin{cases} 1, & \text{if } S = T = 0, \\ 2, & \text{if } S = T = 1, \\ 1, & \text{if } S = T = 2. \end{cases}
$$

Hence, using this and (4-4) we get in total

(4-5)
$$
\boxed{L(1)} \hookrightarrow \boxed{\Delta(0) \twoheadrightarrow L(0)} \quad \boxed{L(0)} \hookrightarrow \boxed{\Delta(1) \twoheadrightarrow L(1)} \quad \boxed{\Delta(2) \cong L(2)}
$$

with $L(\lambda)$ of dimension λ. Next, note that we get from Theorem 3.23 (up to base change)

(4-6)
$$
\mathcal{C}(\mathfrak{u}_0(\mathfrak{sl}_2)) = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathcal{D}(\mathfrak{u}_0(\mathfrak{sl}_2))^{\mathrm{T}} \mathcal{D}(\mathfrak{u}_0(\mathfrak{sl}_2))
$$

which – by (4-5) – actually gives us the indecomposable projectives $P(\lambda)$

$$
\boxed{\Delta(1)} \hookrightarrow \boxed{P(0) \twoheadrightarrow \Delta(0)} \quad \boxed{\Delta(0)} \hookrightarrow \boxed{P(1) \twoheadrightarrow \Delta(1)} \quad \boxed{\Delta(2) \cong P(2)}
$$

Last, (4-6) also shows – by Remark 2.18 – that $\mathfrak{u}_0(\mathfrak{sl}_2)$ is not cellular in the usual sense. However – by Proposition 2.8 – the so-called *core*

(4-7)
$$
\mathfrak{Core}(\mathfrak{u}_0(\mathfrak{sl}_2)) = \textstyle\bigoplus_{\lambda \in X} 1_\lambda \mathfrak{u}_0(\mathfrak{sl}_2) 1_\lambda = 1_0 \mathfrak{u}_0(\mathfrak{sl}_2) 1_0 \oplus 1_1 \mathfrak{u}_0(\mathfrak{sl}_2) 1_1 \oplus 1_2 \mathfrak{u}_0(\mathfrak{sl}_2) 1_2
$$

is a usual cellular algebra. This recovers [BT17a, Theorem 1.2]. It also follows by Proposition 3.28 that $\mathfrak{u}_0(\mathfrak{sl}_2)$ is not semisimple. ▲

Remark 4.9. We stress that our assumption $\chi = 0$ gives (4-p$\mathfrak{sl}_2$). This is crucial since e.g. Lemma 4.7 gives

$$
\mathcal{E}^k\, \mathtt{c}^\lambda_{S,T} \in \textstyle\sum_{j=0}^k \mathbb{K}\, \mathtt{c}^{\lambda+2j}_{S,T}.
$$

Thus, if $\mathcal{E}^p$ would not be zero, then $\lambda + 2p$ would appear in the above sum and (2.1.d) would fail. ▲

4D. **Relative cellularity.** The following is now the main statement in this section.

Theorem 4.10. $\mathfrak{u}_0(\mathfrak{sl}_2)$ is relative cellular with cell datum as in (4-3). □

Proof. Checking (2.1.a). Up to the statement that the $\mathtt{c}^\lambda_{S,T}$ form a basis, this is clear. To see the basis statement use Theorem 4.3.

Checking (2.1.b). This follows since * is the Chevalley anti-involution.

Checking (2.1.c). By construction, the 1_λ's are fixed by *. To see (2.1-c1) note that Lemma 4.4 and Lemma 4.5 show that

$$
1_\mu \mathfrak{u}_0(\mathfrak{sl}_2) 1_\mu = \mathbb{K}\{\mathcal{F}^S 1_\nu \mathcal{E}^S \mid \nu = \mu - 2S\}.
$$

Thus – by Lemma 4.7 – all appearing basis elements in $\mathit{1}_{\mu}\mathfrak{u}_0(\mathfrak{sl}_2)\mathit{1}_{\mu}\,\mathtt{C}^{\lambda}_{S,T}$ are smaller than λ in the order for μ. The rest follows from Lemma 4.5 and Lemma 4.4.

Checking (2.1.d). Directly by using Lemma 4.7 we get

$$\mathtt{C}^{\lambda}_{S,T}\,\mathtt{C}^{\mu}_{U,V} = \mathcal{F}^{S}\mathit{1}_{\lambda}\mathcal{E}^{T}\,\mathcal{F}^{U}\mathit{1}_{\mu}\mathcal{E}^{V} \in r(T,U)\,\mathcal{F}^{S}\mathit{1}_{\lambda}\mathit{1}_{\mu}\mathcal{E}^{V} + \textstyle\sum_{j=0}^{\min(T,U)-1}\mathbb{K}\,\mathcal{F}^{S+T-j}\mathit{1}_{\mu+2(U-j)}\mathcal{E}^{U-j+V}.$$

Thus, (2.1.d) follows since $\mathit{1}_{\lambda}\mathit{1}_{\mu}$ equals $\mathit{1}_{\mu}$ or zero and $\mu+2(U-j)<_{\mathit{1}_{\mu}}\mu$ for $U-j\in\mathsf{F}_p$ and $\mathcal{E}^{U-j+V}=0$ for $U-j\geq p$. ■

4E. **Some consequences.** Similarly as in Example 4.8, we will explain how to recover the representation theory of $\mathfrak{u}_0(\mathfrak{sl}_2)$ for general $p>2$. All of this is of course known, but the point is that we use the general theory of relative cellular algebras to do so.

Proposition 4.11. From the Theorem 4.10 and the theory of relative cellular algebras we obtain the following, where $\lambda\in\mathsf{X}$:

(4.11.a) The cell modules $\Delta(\lambda)$ are of dimension p and isomorphic to baby Verma modules of highest weight λ.

(4.11.b) The simple quotients $L(\lambda)$ of $\Delta(\lambda)$ are of dimension λ and we have

$$\boxed{L(p-\lambda-2)}\hookrightarrow\boxed{\Delta(\lambda)\twoheadrightarrow L(\lambda)}\qquad\boxed{\Delta(p-1)\cong L(p-1)}$$

(4.11.c) The indecomposable projectives $P(\lambda)$ satisfy

$$\boxed{\Delta(p-\lambda-2)}\hookrightarrow\boxed{P(\lambda)\twoheadrightarrow \Delta(\lambda)}\qquad\boxed{P(p-1)\cong \Delta(p-1)}$$

(4.11.d) The algebra $\mathfrak{u}_0(\mathfrak{sl}_2)$ is a non-semisimple, non-cellular algebra whose core (defined as in (4-7)) $\mathfrak{Core}(\mathfrak{u}_0(\mathfrak{sl}_2))$ is cellular in the usual sense. □

Proof. We use all the lemmas from Section 4A and Section 4B. Using these, the general case can be proven verbatim as the $p=3$ case in Example 4.8:

(4.11.a). Clear by construction.

(4.11.b). The first claim follows since

$$\phi^{\lambda}(\mathcal{F}^{S}\mathit{1}_{\lambda},\mathcal{F}^{S}\mathit{1}_{\lambda}) = \begin{cases}(S!)^2\binom{\lambda}{S}, & \text{if } S=T,\\ 0, & \text{if } S\neq T.\end{cases}$$

The second claim follows then from (4.11.a).

(4.11.c). By using (4.11.b) and Theorem 3.23.

(4.11.d). Observe that (4.11.b) shows – by Proposition 3.28 – that $\mathfrak{u}_0(\mathfrak{sl}_2)$ is non-semisimple, while (4.11.c) – by Remark 2.18 – shows that $\mathfrak{u}_0(\mathfrak{sl}_2)$ is not cellular in the usual sense. The last claim follows from Theorem 4.10 and Proposition 2.8. ■

This resembles the known representation theory of $\mathfrak{u}_0(\mathfrak{sl}_2)$ from the theory of relative cellular algebras.

Remark 4.12. The case of the small quantum group $\mathfrak{u}_q(\mathfrak{sl}_2)$ for q being a complex, primitive $2l^{\text{th}}$ root of unity with $l>2$ works – by carefully keeping track of the quantum numbers – mutatis mutandis as above. Details are omitted. ▲

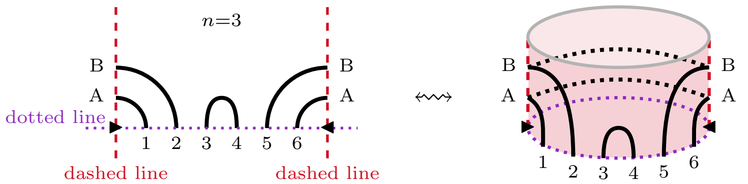
Further directions 4.13. Having (4.11.d), it is tempting to ask whether one can extend the setting of [BT17b] and [BT17a]. However, we stress that our above basis is too "naive" to generalize to higher rank cases and certainly not the relative analog of the basis of $\mathfrak{Core}(\mathfrak{u}_0(\mathfrak{sl}_2))$ constructed in [BT17a, Theorem 4.6]. ▲

5. An extended example II: The annular arc algebra

Throughout, fix $n \in \mathbb{Z}_{>0}$. The purpose of this section is to discuss the relative cellularity of the annular arc algebra $\mathfrak{A}rc_n^{\mathrm{ann}}$ in detail, with Theorem 5.17 being the main result.

5A. **The arc algebra in an annulus.** We will define $\mathfrak{A}rc_n^{\mathrm{ann}}$ very much in the spirit of the usual, type A arc algebra $\mathfrak{A}rc_n$ (see e.g. [Kho02] or [BS11]), but using a TQFT as in [APS04], where $\mathfrak{A}rc_n^{\mathrm{ann}}$ appears implicitly. Consequently, all the definitions below are adaptations of the corresponding notions for $\mathfrak{A}rc_n$ to the annulus, where we keep the following illustration in mind:

(5-annulus)



(In (5-annulus), note that the annulus is topologically a cylinder, a perspective which we use silently throughout.) Readers familiar with $\mathfrak{A}rc_n$ can immediately check 5.Merge and 5.Split in addition to (5-annulus) before reading the definitions.

Remark 5.1. We try to keep the involved notions as easy as possible in this paper. But the below can be extended to the setting of a *generalized annular arc algebra*. This is similar in flavor to the usual setup in e.g. [BS11, Section 5]. ▲

5B. **Combinatorics of annular arc diagrams.** We start by defining the necessary combinatorial data. Hereby we closely follow the exposition in the non-annular case from [BS11, Section 2] or [EST17, Section 3].

Definition 5.2. A (balanced) *weight* (of rank n) is a tuple $\lambda = (\lambda_i) \in \{\vee, \wedge\}^{2n}$ with n symbols $\vee$ and n symbols $\wedge$. The set of weights is denoted by X. ▲

Simplifying notation, an example of a weight of rank 2 is $\lambda = \vee\wedge\wedge\vee$.

Let $\mathbb{S}^1$ denote the 1-sphere. The *dotted line* is topologically $\mathbb{S}^1 \times \{0\}$ smoothly embedded in $\mathbb{R}^2 \times \{0\}$ together with a choice of an orientation (which will always be anticlockwise in illustrations), two distinct points $\blacktriangleright, \blacktriangleleft$ and $2n$ discrete points, called *vertices*, in the segment $[\blacktriangleright, \blacktriangleleft]$ between $\blacktriangleright$ and $\blacktriangleleft$. We number the vertices in order from 1 to $2n$, reading along the chosen orientation. We view the dotted line as being the bottom (or top) boundary of $\mathbb{S}^1 \times [0,1]$ (or $\mathbb{S}^1 \times [0,-1]$) smoothly embedded in $\mathbb{R}^3$, with orientation compatible with the one of the dotted line. Similarly, the *dashed lines* are $\{\blacktriangleright\} \times [0,\pm 1]$ and $\{\blacktriangleleft\} \times [0,\pm 1]$, see again in $\mathbb{S}^1 \times [0,\pm 1]$. Note that each $\lambda = (\lambda_i) \in \mathsf{X}$ gives a labeling of the vertices of the dotted line by putting λ_i at the i^{th} vertex.

Definition 5.3. A(n annular) *cup diagram* S (of rank n) is a collection $\{\gamma_1, \dots, \gamma_n\}$ of smooth embeddings of $[0,1]$ into $\mathbb{S}^1 \times [0,-1]$, called *arcs*, such that:

(5.3.a) The arcs are pairwise non-intersecting and have only one critical point.

(5.3.b) There is a 1:1 correspondence between the vertices of the dotted line and the boundary points of arcs, identifying the two sets.

(5.3.c) The arcs cut the dashed lines transversely and each dashed line at most once.

Similarly, a(n annular) *cap diagram* $T^\star$ is defined inside $\mathbb{S}^1 \times [0,1]$.

Observing that (5.3.b) and (5.3.c) imply that each arc either stays within the region $[\blacktriangleright, \blacktriangleleft] \times [0, \pm 1]$ or goes around the cylinder once, we can say that an arc is of *type stay* or *type around*. Similarly, if all arcs of a cup (or cap) diagram are of type stay, then we say that the cup (or cap) diagram is of *type stay*. ▲

Combinatorially speaking, we consider arcs to be equal if their endpoints connect the same vertices on the dotted line and they are of the same type, and the corresponding equivalence classes are still called cup and cap diagrams. We work with these throughout, and illustrate them as exemplified in (5-1). We call the corresponding arcs *cups* and *caps*, and we usually denote them by $\cup$ respectively by $\cap$.

We note that cup (or cap) diagrams of type stay are those appearing for $\mathfrak{A}rc_n$, while all others are new in the annular setting.

Definition 5.4. An *orientated cup diagram* $S\lambda$ is a pair of a cup diagram and a weight λ such that the weight induces an orientation on the arcs of S (seen topologically). An *orientated cap diagram* $\lambda T^\star$ is defined verbatim.

For $\lambda \in \mathsf{X}$ we denote by $\mathsf{M}(\lambda)$ the set of all oriented cup diagrams of the form $S\lambda$. ▲

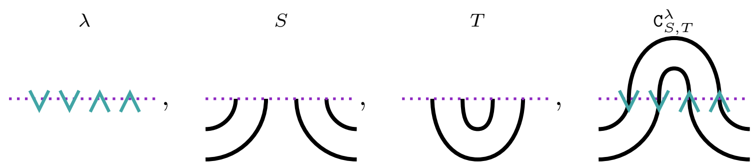
Note that we can swap the cylinders $\mathbb{S}^1 \times [0,-1] \rightleftarrows \mathbb{S}^1 \times [0,1]$ by reflecting along the $(x,y,0)$-plane in $\mathbb{R}^3$. This induces an involution * turning a cup S into a cap diagram $S^\star$, and vice versa. Clearly, $(S^\star)^\star = S$, and – by convention – $(S\lambda)^\star = \lambda S^\star$ and $(\lambda S^\star)^\star = S\lambda$.

Definition 5.5. A(n annular) *circle diagram* $ST^\star$ (of rank n) is obtained from a cup diagram S and a cap diagram $T^\star$ (both of rank n) by stacking $T^\star$ on top of S, inducing a corresponding diagram in $\mathbb{S}^1 \times [-1,1]$.

An *oriented circle diagram* is built from an oriented cup $S\lambda$ and cap diagram $\lambda T^\star$ for the same weight λ. We denote such diagrams by $\mathtt{c}^\lambda_{S,T}$, and we say that the circle diagram $ST^\star$ is *associated to* $\mathtt{c}^\lambda_{S,T}$. ▲

Similar as cup and cap diagrams are built from arcs, circle diagrams are collections of (up to n) *circles* $\mathcal{C}$, with "circle" understood in the evident way.

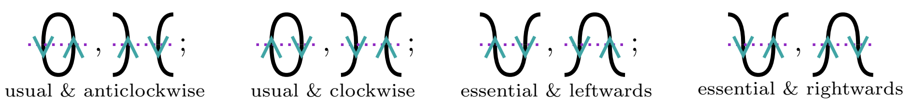
All the above is summarized in (5-1) below.

(5-1) 

Definition 5.6. A circle $\mathcal{C}$ in a circle diagram $ST^\star$ is called *essential* if it induces a non-trivial element in $\pi_1(\mathbb{S}^1 \times [-1,1])$, and *usual* otherwise.

For an oriented circle diagram $S\lambda T^\star$, any circle $\mathcal{C}$ gets an induced orientation. Thus, we can say a usual circle is *anticlockwise* or *clockwise* (oriented), while essential circles are *leftwards* or *rightwards* (oriented). ▲

The picture illustrating Definition 5.6 is:

(5-2) 

(As in (5-2), we say e.g. *usual & clockwise* for short.)

5C. **The annular arc algebra.** We define the annular arc algebra first as a vector space, and explain the multiplication afterwards.

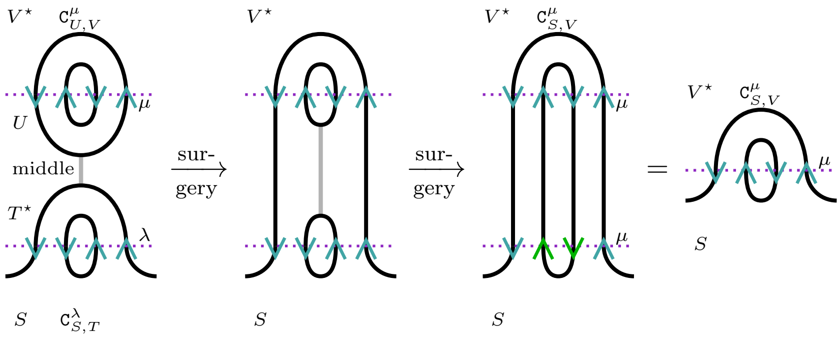
Definition 5.7. The *annular arc algebra* $\mathfrak{A}rc_n^{\mathrm{ann}}$ (of rank n) is

$$\mathfrak{A}rc_n^{\mathrm{ann}} = \mathbb{K}\{\mathtt{C}_{S,T}^{\lambda} \mid \lambda \in \mathsf{X},\, S,T \in \mathsf{M}(\lambda)\},$$

i.e. the free vector space on basis given by all oriented circle diagrams (of rank n). The multiplication is given by a so-called *surgery* as explained below. ▲

Before we define the multiplication by a *surgery procedure*, here a prototypical example, each step called a(n oriented) *stacked diagram*:

(5-3)



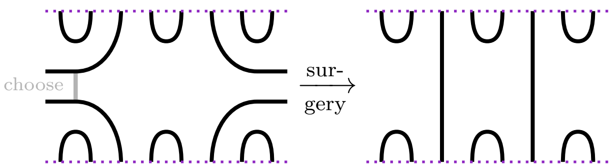
(In our notation, left multiplication is given by concatenation from the bottom.)

To define the multiplication $\mathrm{Mult}\colon \mathfrak{A}rc_n^{\mathrm{ann}} \otimes \mathfrak{A}rc_n^{\mathrm{ann}} \to \mathfrak{A}rc_n^{\mathrm{ann}}$ it suffices to explain it on two basis elements $\mathtt{C}_{S,T}^{\lambda}$ and $\mathtt{C}_{U,V}^{\mu}$, and extend linearly. The multiplication of such basis elements is defined as follows.

(5.stack) We let $\mathtt{C}_{S,T}^{\lambda}\mathtt{C}_{U,V}^{\mu} = 0$ unless $T = U$. Otherwise, put the circle diagram associated to $\mathtt{C}_{U,V}^{\mu}$ on top of the one associated to $\mathtt{C}_{S,T}^{\lambda}$, producing a stacked diagram having $T^{\star}U$ in the middle, cf. (5-3).

(5.surgery) For the stacked diagram perform inductively a surgery procedure by picking any (note that there is a choice involved) $\cup$-$\cap$ pair which is available, meaning that the $\cup$ and the $\cap$ can be connected without crossing any other arc, and replace it locally via:

(5-4)



(5.reorient) In each step of (5.surgery) we replace the resulting stacked diagrams by a sum of (oriented) stacked diagrams as explained below.

(5.collapse) Last, collapse the resulting stacked diagrams to circle diagrams as illustrated on the right in (5-3).

Observing that each step of (5.surgery) either merges two circles into one, or splits one circle into two, we define how to reorient diagrams as follows. In all cases, we say "orient the result" meaning to put the corresponding orientation locally on the stacked diagram after applying (5.surgery), leaving all non-involved parts with the same orientation.

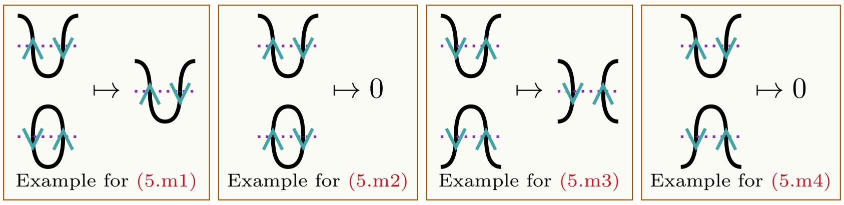
5.Merge. Assume that two circles are merge into one.

(5.m1) If one of the circles is usual & anticlockwise, then orient the result with the orientation induced by the other circle.

(5.m2) If one the circles is usual & clockwise and the other is not usual & anticlockwise, then the result is zero.

(5.m3) If one the circles is essential & leftwards and the other is essential & rightwards, then orient the result clockwise.

(5.m4) Otherwise, the result is zero.


Example for (5.m1) Example for (5.m2) Example for (5.m3) Example for (5.m4)

5.Split. Assume that one circle is split into two.

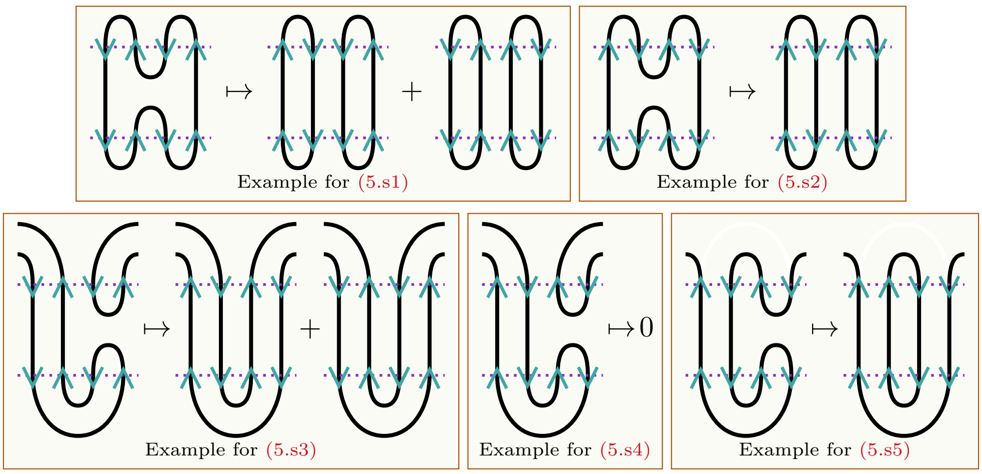
(5.s1) If the circle is usual & anticlockwise and splits into two usual circles $\mathcal{C}_1$ and $\mathcal{C}_2$, then take the sum of two copies of the result. In one summand orient $\mathcal{C}_1$ clockwise and $\mathcal{C}_2$ anticlockwise, in the other swap the roles.

(5.s2) If the circle is usual & clockwise and splits into two usual circles, then orient both circles in the result clockwise.

(5.s3) If the circle is usual & anticlockwise and splits into two essential circles $\mathcal{C}_1$ and $\mathcal{C}_2$, then take the sum of two copies of the result. In one summand orient $\mathcal{C}_1$ leftwards and $\mathcal{C}_2$ rightwards, in the other swap the roles.

(5.s4) If the circle is usual & clockwise and splits into two essential circles, then the result is zero.

(5.s5) If the circle is essential, then orient the resulting usual circle clockwise while keeping the orientation of the resulting essential circle.


Example for (5.s1) Example for (5.s2) Example for (5.s3) Example for (5.s4) Example for (5.s5)

We leave it to the reader to check that 5.Merge and 5.Split exhaust all possible configurations. Before giving some examples, let us state the first non-trivial result about $\mathfrak{Arc}_n^{\mathrm{ann}}$, where the well-definedness is the crucial statement.

Proposition 5.8. The multiplication is well-defined, i.e. independent of all involved choices. This turns $\mathfrak{Arc}_n^{\mathrm{ann}}$ into an associative, unital, finite-dimensional algebra with

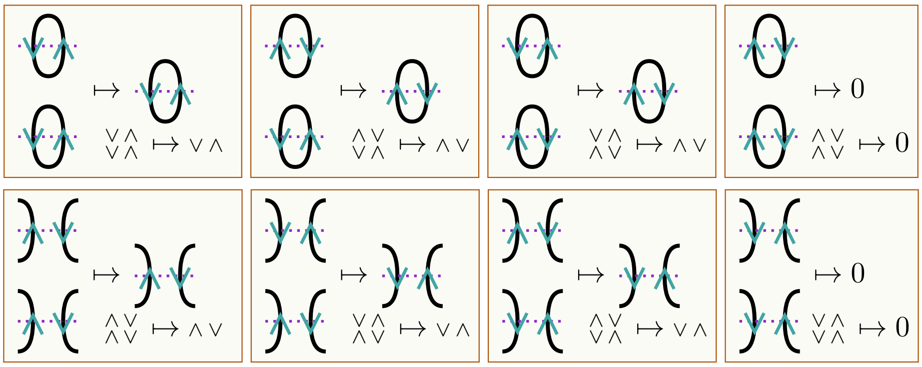
$$\mathsf{E} = \{\mathsf{c}_{S,S}^{\lambda} \mid S\lambda S^{\star} \text{ contains only usual \& anticlockwise } \mathcal{C}\text{s}\} \tag{5-5}$$

being a complete set of pairwise orthogonal idempotents. □

Proof. With the well-definedness as an exception, the statements are easy to verify. That the multiplication is well-defined follows by identifying the annular arc algebra with a topological algebra obtained via a TQFT. We omit the details which in the usual setting are explained in [EST17], and only note that the underlying TQFT which we use is based on [APS04]. For further details about this TQFT see e.g. [Rob13, Section 2], [GLW15, Section 4.2] or [BPW16, Section 2.4]. ■

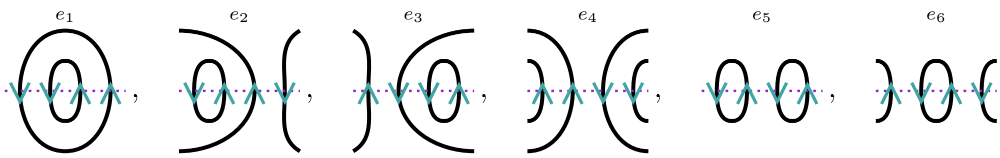
Further directions 5.9. The TQFT used in the proof of Proposition 5.8 originates in the context of versions of annular link homologies, see e.g. the references above. It would be interesting to know a connection from our work on $\mathfrak{Arc}_n^{\mathrm{ann}}$ to those homologies. ▲

Example 5.10. Here the multiplication for symmetric pictures in case $n = 1$:



Note the changed roles of the weights. ▲

Example 5.11. The list of the idempotents from (5-5) in case $n = 2$ is



(For later use, cf. Example 5.24, we denote them by e_i for $i = 1, \dots, 6$.) ▲

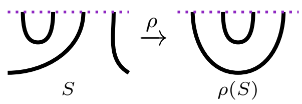
Further directions 5.12. $\mathfrak{Arc}_n^{\mathrm{ann}}$ (and its generalization as in Remark 5.1) also appears conjecturally, with different sign conventions, in [AN16, Section 5.3]. ▲

5D. **Relative cellularity: The cell datum.** Let us now give the relative cell datum.

First, as already indicated by our notation in Section 5B, the set X is the set of weights, while the sets $\mathsf{M}(\lambda)$ are those cup diagrams S such that $S\lambda$ is oriented. The map C is then given by the defined basis elements $\mathsf{c}_{S,T}^{\lambda}$. The anti-involution $\star$ is given by reflection.

Furthermore – by Proposition 5.8 – we let E be as in (5-5), and we can associate to a cup diagram S the idempotent $\varepsilon_S = \mathsf{c}_{S,S}^{\lambda} \in \mathsf{E}$. This in turn defines the map $\epsilon(S) = \varepsilon_S$.

To define the partial orders $<_{\varepsilon_S}$ with respect to the idempotents in E, note that there is a rotation map $\rho\colon \mathsf{X} \to \mathsf{X}$ given by rotating rightwards. This is formally done by renumbering the vertices on the dotted line to $2, 3, \cdots, 2n, 1$. The same is done for cup diagrams, e.g.:

(5-6)
$$\begin{matrix} \wedge\vee\wedge\vee \xrightarrow{\rho} \vee\wedge\vee\wedge \\ \wedge\vee\vee\wedge \xrightarrow{\rho} \wedge\wedge\vee\vee \end{matrix},$$


We note two lemmas whose (very easy) proofs we omit.

Lemma 5.13. The map ρ defined on the basis as $\rho(\mathsf{C}^{\lambda}_{S,T}) = \mathsf{C}^{\rho(\lambda)}_{\rho(S),\rho(T)}$ defines an algebra automorphism of $\mathfrak{A}rc^{\mathrm{ann}}_n$. ■

Lemma 5.14. For each cup diagram S there is $k \in \mathbb{Z}_{\geq 0}$ such that the cup diagram $\rho^k(S)$ is of type stay. ■

The set X has a partial order $\prec_{\mathfrak{A}rc_n}$ generated by saying that an ordered pair $\vee\wedge$ swapped to $\wedge\vee$ creates a smaller element of X. (This is actually the partial order for $\mathfrak{A}rc_n$, cf. [BS11, Section 2].) Starting from this partial order we will define – by using Lemma 5.14 – our partial orders using the rotation ρ.

Definition 5.15. Let S be a cup diagram and $\lambda, \mu \in \mathsf{X}$. Let $k \in \mathbb{Z}_{\geq 0}$ be minimal such that $\rho^k(S)$ is of type stay. Then we define $\mu <_{\varepsilon_S} \lambda$ if $\rho^k(\mu) \prec_{\mathfrak{A}rc_n} \rho^k(\lambda)$. ▲

For example, $\wedge\vee\vee\wedge <_{\varepsilon_S} \wedge\vee\wedge\vee$, but $\wedge\vee\wedge\vee <_{\varepsilon_{\rho(S)}} \wedge\vee\vee\wedge$ for S as in (5-6).

Now – by Definition 5.15 – we let $\mathsf{O} = \{<_{\varepsilon_S} | \; S \text{ is a cup diagram}\}$, and get

(5-cell-datum)
$$(\mathsf{X}, \mathsf{M}, \mathsf{C}, {}^{\star}, \mathsf{E}, \mathsf{O}, \epsilon)$$

which will be our relative cell datum.

The main ingredient to prove relative cellularity is the following which is similar to [BS11, Theorem 3.1], but more involved to prove. Its proof appears in Section 5I below.

Theorem 5.16. Let $\lambda, \mu \in \mathsf{X}$, $S, T \in \mathsf{M}(\lambda)$ and $U, V \in \mathsf{M}(\mu)$. Then

$$\mathsf{C}^{\lambda}_{S,T}\mathsf{C}^{\mu}_{U,V} = \begin{cases} 0, & \text{if } T \neq U, \\ r(\mathsf{C}^{\lambda}_{S,T}, U)\mathsf{C}^{\mu}_{U,V} + (\dagger), & \text{if } T = U \text{ and } V \in \mathsf{M}(\mu), \\ (\dagger), & \text{otherwise,} \end{cases}$$

with $r(\mathsf{C}^{\lambda}_{S,T}, U) \in \{0, 1\} \subset \mathbb{K}$, $(\dagger) \in \mathfrak{A}rc^{\mathrm{ann}}_n(<_{\varepsilon_V} \mu)$ and $\varepsilon_S(\dagger) = (\dagger) = (\dagger)\varepsilon_V$. □

This in turn implies the relative cellularity of the annular arc algebra.

Theorem 5.17. $\mathfrak{A}rc^{\mathrm{ann}}_n$ is relative cellular with cell datum as in (5-cell-datum). □

Proof. *(2.1.a)*. The sets X and $\mathsf{M}(\lambda)$ are clearly finite, and the assignment C gives – by definition – an injective map with image forming a basis of $\mathfrak{A}rc^{\mathrm{ann}}_n$.

(2.1.b). Clearly, * is an anti-involution with $(\mathsf{C}^{\lambda}_{S,T})^{\star} = \mathsf{C}^{\lambda}_{T,S}$.

(2.1.c). All statements about the idempotents and the mapping ϵ are – by e.g. Proposition 5.8 – immediate except (2.1-c1). For (2.1-c1) we note that $\varepsilon\mathfrak{A}rc^{\mathrm{ann}}_n\varepsilon\mathsf{C}^{\lambda}_{S,T}$ is zero unless $\varepsilon = \varepsilon_S$. In this case $\varepsilon\mathfrak{A}rc^{\mathrm{ann}}_n\varepsilon$ is spanned by elements of the form $\mathsf{C}^{\mu}_{S,S}$ for $\mu \in \mathsf{X}$. The multiplication $\mathsf{C}^{\mu}_{S,S}\mathsf{C}^{\lambda}_{S,T}$ will be a merge in each step and the only non-trivial operation is that some circles in $ST^{\star}$ are reoriented from anticlockwise to clockwise. However – by Lemma 5.35 below – this will decrease the weight with respect to both, $<_{\varepsilon_S}$ and $<_{\varepsilon_T}$.

(2.1.d). We note that Theorem 5.16 is a stronger version of (2.1.d). ■

5E. **Further properties.** By Theorem 5.17 we can use the notions from Section 3 regarding simples, cell and indecomposable projective $\mathfrak{A}rc_n^{\mathrm{ann}}$-modules.

Proposition 5.18. Let $\lambda,\mu \in \mathsf{X}$ and $S \in \mathsf{M}(\lambda)$, $T \in \mathsf{M}(\mu)$ such that $\varepsilon_S = \mathtt{C}_{S,S}^{\lambda}$ and $\varepsilon_T = \mathtt{C}_{T,T}^{\mu}$. Then the following hold.

(5.18.a) $[\Delta(\lambda) : L(\mu)] = 1$ if and only if $T\lambda$ is oriented, otherwise it is zero.

(5.18.b) The projective $P(\lambda)$ has a filtration by cell modules of the form $\Delta(\nu)$ such that $S\nu$ is oriented. Further, it has a filtration by 2^n cell modules, each occurring once.

(5.18.c) $[P(\lambda) : L(\mu)]$ can be computed by counting the number of orientations of $ST^{\star}$, with each orientation ν giving the occurrence of $L(\mu)$ in $\Delta(\nu)$ inside the cell module filtration given by (5.18.b). □

Proof. *(5.18.a).* This follows immediately by noting that the basis elements of $\Delta(\lambda)$ are compatible with the choice of primitive idempotents, with exactly one of the idempotents acting as 1 on a given basis element and all others acting by 0.

(5.18.b). The first statement follows by construction of the cell filtration in Proposition 3.19. The second statement follows since the number of orientations for S is exactly n.

(5.18.c). By combining (5.18.a) and (5.18.b). ■

Remark 5.19. Note that – by the proof of (5.18.a) – it also follows that simple modules always have dimension one. On the other hand, a cell module $\Delta(\lambda)$ has dimension equal to the number of cup diagrams S such that $S\lambda$ is oriented. Thus, its dimension is always > 1. Furthermore, (5.18.b) implies that $P(\lambda)$ is also always different from $\Delta(\lambda)$. To summarize: No cell module is simple or projective. ▲

Proposition 5.20. The algebra $\mathfrak{A}rc_n^{\mathrm{ann}}$ is a non-semisimple Frobenius algebra of infinite global dimension. □

Proof. A bilinear form $\sigma\colon \mathfrak{A}rc_n^{\mathrm{ann}} \otimes \mathfrak{A}rc_n^{\mathrm{ann}} \to \mathbb{K}$ is given by $\sigma(\mathtt{C}_{S,T}^{\lambda}, \mathtt{C}_{U,V}^{\mu}) = 0$ for $S \neq V$, and otherwise $\sigma(\mathtt{C}_{S,T}^{\lambda}, \mathtt{C}_{U,V}^{\mu})$ is set to be the coefficient of $\mathtt{C}_{S,S}^{\nu}$ in the product $\mathtt{C}_{S,T}^{\lambda}\mathtt{C}_{U,V}^{\mu}$, where ν is chosen such that all circles in $S\nu S^{\star}$ are oriented clockwise. Associativity and non-degeneracy can be shown using the same TQFT methods as in [EST17], using the TQFT as in the proof of Proposition 5.8.

From the dimension observations in Remark 5.19 it follows that $\mathfrak{A}rc_n^{\mathrm{ann}}$ is non-semisimple. Last, recall that a Frobenius algebra has finite global dimension if and only if it is semisimple. Thus, $\mathfrak{A}rc_n^{\mathrm{ann}}$ is of infinite global dimension. ■

Remark 5.21. The Frobenius property in Proposition 5.20 can be proven directly using combinatorics. While associativity of σ follows immediately, the non-degeneracy can be checked by carefully looking at products of the form $\mathtt{C}_{S,T}^{\lambda}\mathtt{C}_{T,S}^{\mu}$ and noting that the surgeries can be ordered so that merges are performed first followed by splits. Thus, for a given weight λ, the μ can be chosen appropriately so that all circles, after performing the merges, are usual & clockwise and then the splits will all create usual & clockwise circles, giving the non-degeneracy of σ. ▲

Proposition 5.22. $\mathcal{C}(\mathfrak{A}rc_n^{\mathrm{ann}})$ is positive semidefinite with determinant zero. □

Proof. By Corollary 3.25 it remains to check that the Cartan matrix is not of full rank.

The case $n = 1$ is done explicitly in Example 5.23 below.

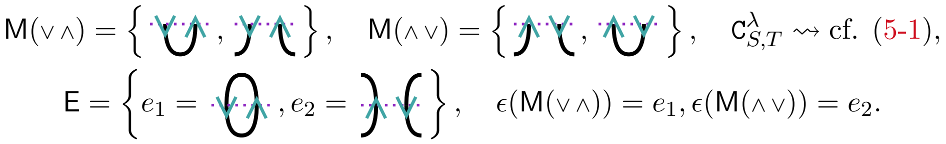
For the case $n > 1$, let S be the cup diagram having only arcs of type stay with one arc connecting vertices 1 and $2n$ and arcs connecting $2i$ and $2i+1$ for $1 \leq i \leq (n-1)$. Let

$\varepsilon_S = S\lambda S^\star$ for $\lambda = \vee(\vee\wedge)(\vee\wedge)\cdots(\vee\wedge)\wedge$. The multiplicity $[P(\lambda) : L(\mu)]$ is – by (5.18.c) – obtained by counting the number of possible orientations of the diagram $ST^\star$, where T is the unique diagram such that $T\mu T^\star$ is a primitive idempotent. Note that this is 2^m for m being the number of circles in $ST^\star$. Next, the number of such orientations is the same as the number of orientations of $\rho^2(S)T^\star$. This holds true since $\rho^2(S)$ connects the same vertices as S, just with arcs that are not of type stay. Thus, $[P(\lambda) : L(\mu)] = [P(\rho^2(\lambda)) : L(\mu)]$ and with the assumption $n > 1$ it holds that $\rho^2(\lambda) \neq \lambda$. In total, the matrix $\mathcal{C}(\mathfrak{A}rc_n^{\mathrm{ann}})$ has two equal columns. ■

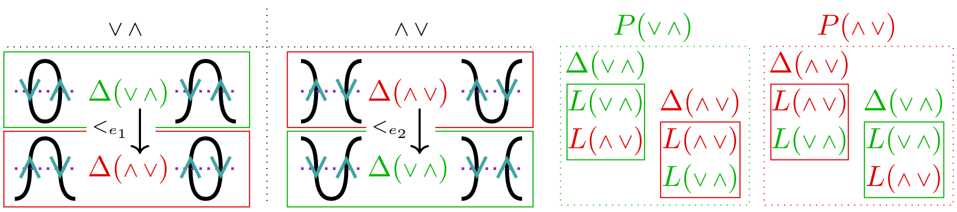
5F. **Low rank examples.** Let us discuss the cases $n = 1$ and $n = 2$ in detail. This will be very much as in Section 2E, whose notions we recommend to recall.

Example 5.23. Let $n = 1$. Then the relative cell datum of $\mathfrak{A}rc_1^{\mathrm{ann}}$ is as follows.

$$\mathsf{X} = \{\boxed{\wedge\vee} <_{e_1} \boxed{\vee\wedge}\} = \{\boxed{\vee\wedge} <_{e_2} \boxed{\wedge\vee}\}, \quad {}^\star \rightsquigarrow \text{reflect diagrams},$$



Now, as for the usual arc algebra, the indecomposable projectives $P(\vee\wedge) = \mathfrak{A}rc_1^{\mathrm{ann}}e_1$ and $P(\wedge\vee) = \mathfrak{A}rc_1^{\mathrm{ann}}e_2$ are given by fixing (in our notation) the top shape. In contrast, the cell modules $\Delta(\vee\wedge)$ and $\Delta(\wedge\vee)$ are given by fixing the weight, and we get



Note that looking at the bottom picture determines the action of the primitive idempotents e_1 and e_2, and thus the simple module as illustrated.

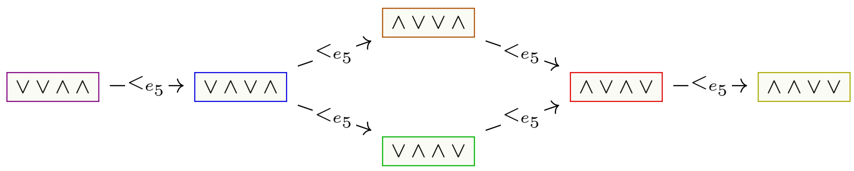
Last, the above gives us the Cartan matrix $\mathcal{C}(\mathfrak{A}rc_1^{\mathrm{ann}}) = \left(\begin{smallmatrix} 2 & 2 \\ 2 & 2 \end{smallmatrix}\right)$, showing that $\mathfrak{A}rc_1^{\mathrm{ann}}$ is not cellular in the usual sense. ▲

Example 5.24. Let $n = 2$. We are now going to explain how the relative cellular datum of $\mathfrak{A}rc_2^{\mathrm{ann}}$ looks like. The relative cellular datum will be very much in the spirit as in Example 5.23, with partial orderings relative to the idempotents in Example 5.11. But since the algebra $\mathfrak{A}rc_2^{\mathrm{ann}}$ is of dimension 108, we will only highlight some features by focussing on e_2 and e_5.

First, we have $\mathsf{X} = \{\boxed{\vee\vee\wedge\wedge}, \boxed{\vee\wedge\vee\wedge}, \boxed{\wedge\vee\vee\wedge}, \boxed{\vee\wedge\wedge\vee}, \boxed{\wedge\vee\wedge\vee}, \boxed{\wedge\wedge\vee\vee}\}$ as the set of weights. As explained in Section 5D, the partial orderings for the idempotents are obtained from the usual one (which is (5-8) in this case) by 'rotation of the cylinder', e.g.

$$\begin{tikzcd}[column sep=small]
 & & \boxed{\vee\vee\wedge\wedge} \arrow[dr, "<_{e_2}"] & & \\
\boxed{\vee\wedge\wedge\vee} \arrow[r, "<_{e_2}"] & \boxed{\wedge\vee\wedge\vee} \arrow[ur, "<_{e_2}"] \arrow[dr, "<_{e_2}"'] & & \boxed{\vee\wedge\vee\wedge} \arrow[r, "<_{e_2}"] & \boxed{\wedge\vee\vee\wedge} \\
 & & \boxed{\wedge\wedge\vee\vee} \arrow[ur, "<_{e_2}"'] & &
\end{tikzcd} \tag{5-7}$$

(5-8)

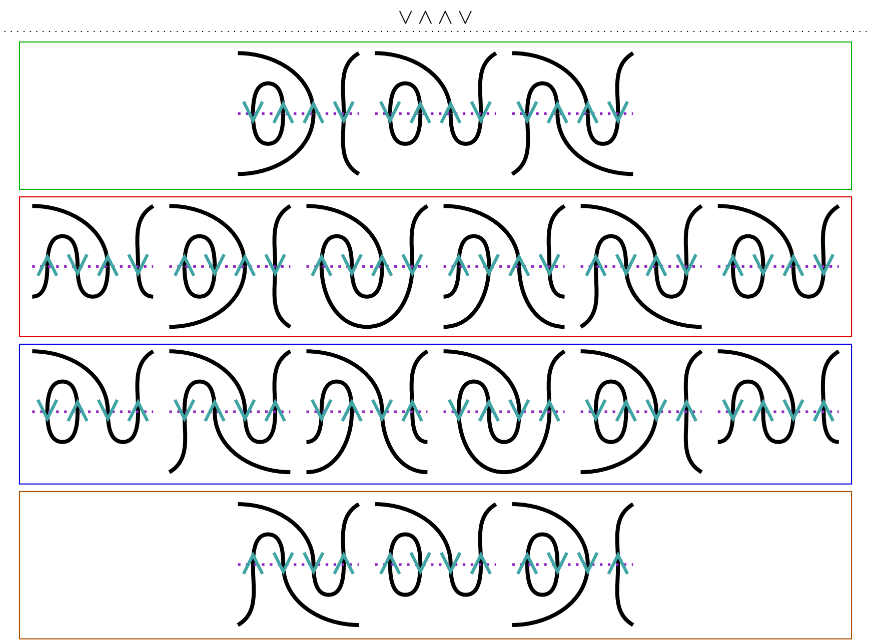


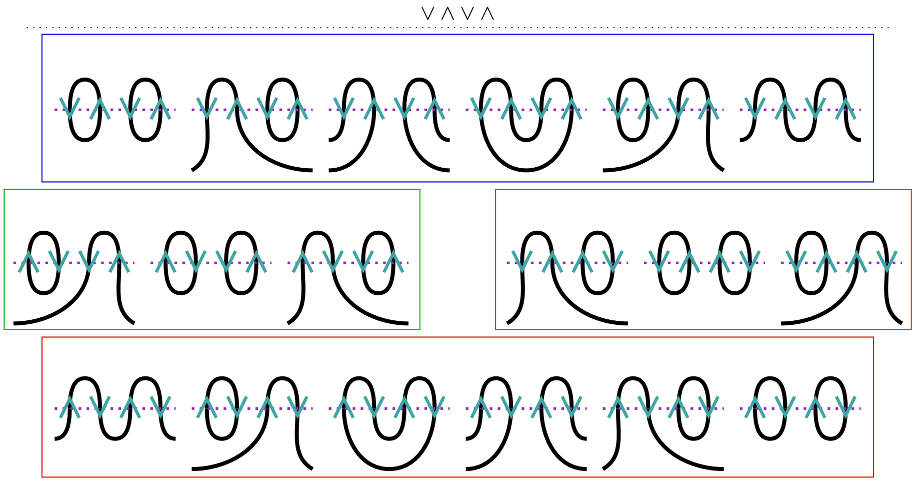
The other partial orderings are similar, but rotated.

Now, the relative cell datum is

$$\begin{gathered}\mathsf{X} \rightsquigarrow \text{cf. (5-7)}, \quad {}^{\star} \rightsquigarrow \text{reflect diagrams}, \quad \mathsf{M} \rightsquigarrow \text{cf. Example 5.24}, \\ \mathsf{c}^{\lambda}_{S,T} \rightsquigarrow \text{cf. (5-1)}, \quad \mathsf{E} \rightsquigarrow \text{cf. Example 5.11} \quad \epsilon \rightsquigarrow \text{cf. Example 5.24}.\end{gathered}$$

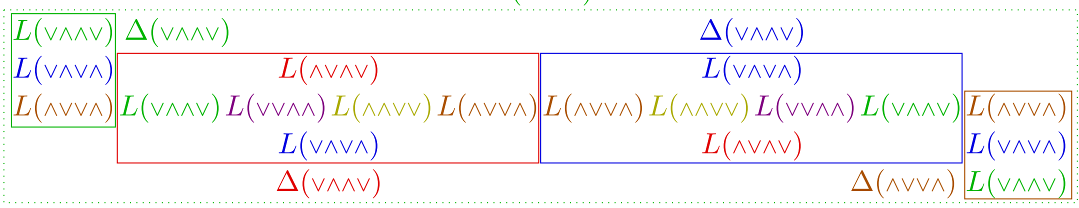
Having these, the cell modules are



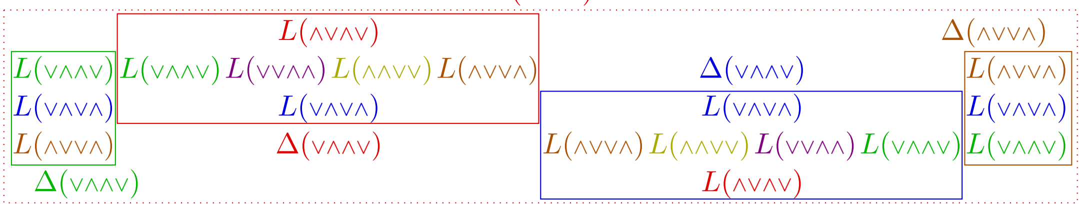


which are ordered as in (5-7) and (5-8). Next, the indecomposable projectives are

$P(\vee\wedge\wedge\vee)$



$P(\vee\wedge\wedge\vee)$



(Note: From $n = 3$ onwards the $P(\lambda)$'s are not of the same size anymore. That is, $\mathfrak{A}rc_3^{\mathrm{ann}}$ is of dimension 1664 with $P(\lambda)$'s being of dimension 80 or 88.) Last, by the above we see that the Cartan matrix is (up to base change)

$$C(\mathfrak{A}rc_2^{\mathrm{ann}}) = \begin{pmatrix} 4 & 2 & 2 & 4 & 2 & 4 \\ 2 & 4 & 4 & 2 & 4 & 2 \\ 2 & 4 & 4 & 2 & 4 & 2 \\ 4 & 2 & 2 & 4 & 2 & 4 \\ 2 & 4 & 4 & 2 & 4 & 2 \\ 4 & 2 & 2 & 4 & 2 & 4 \end{pmatrix}$$

which again shows that $\mathfrak{A}rc_2^{\mathrm{ann}}$ is not a usual cellular algebra. ▲

5G. **Some concluding comments.** A few potential generalizations regarding relative cellularity of $\mathfrak{A}rc_n^{\mathrm{ann}}$ are:

Further directions 5.25. Everything can be done in the graded setup as well with the algebra $\mathfrak{A}rc_n^{\mathrm{ann}}$ having an analogous grading as $\mathfrak{A}rc_n$. In particular, it makes sense to define the notion of a *graded, relative cellular algebra*, generalizing [HM10, Definition 2.1]. ▲

Further directions 5.26. $\mathfrak{A}rc_n$ was originally defined to construct tangle invariants associated to Khovanov homology [Kho02]. Similarly, so-called *web algebras* appear in the construction of tangle invariants associated to Khovanov–Rozansky homologies. These web algebras are also known to be usual cellular algebras, see [MPT14, Corollary 5.21], [Tub14, Theorem 4.22] and [Mac14, Theorem 7.7]. Building on [QR15], it should be possible to defined annular variants, and the question whether these are relative cellular arises. ▲

Further directions 5.27. One could also define annular versions of the *type* D *arc algebra* as in [ES16b], [ES16a] or [ETW16]. This algebra is again cellular in the usual sense, see [ES16b, Corollary 7.3], and the question about relative cellularity again arises. ▲

5H. **Relative cellularity: Technicalities.** For the proof of Theorem 5.16 we need some more control over cups and caps, necessitating a number of definitions and lemmas.

Definition 5.28. Let $\lambda \in \mathsf{X}$ and S be a cup diagram such that $S\lambda$ is oriented. Assume that we have the following local situations.

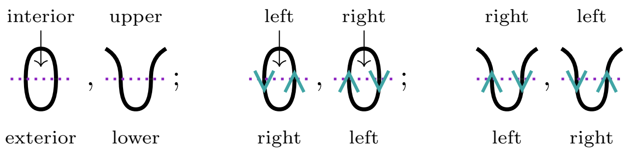
(5-9)


Then we call such cups or caps *anticlockwise* and *clockwise*, as indicated. ▲

Comparing to (5-2), cups and caps in usual circles are always of the corresponding orientation. Moreover, cups in essential & rightwards and caps in essential & leftwards circles are clockwise, and vice versa.

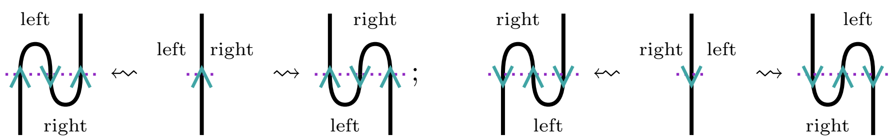
Definition 5.29. Let $\mathcal{C}$ be a circle in a circle diagram $ST^{\star}$. Then $\mathbb{S}^1 \times [-1,1] \setminus \mathcal{C}$ has two connected components. For a usual circle the connected component containing the boundary of $\mathbb{S}^1 \times [-1,1]$ is called the *exterior* of $\mathcal{C}$, the other is called the *interior*. For an essential circle the one containing the boundary $\mathbb{S}^1 \times \{1\}$ is called the *upper* (half), the other is called the *lower* (half).

Here the picture which illustrates these notions:

(5-10)


As in (5-10), if furthermore a *small circle* $\mathcal{C}$ (i.e. circles built from one cup and one cap only) is endowed with an orientation in $\mathsf{C}^{\lambda}_{S,T}$, then we distinguish between a *right* and a *left side of* $\mathcal{C}$ by using the orientation.

For more general circles we use repeatedly

(5-zigzag)


to define the notions *right* and *left side of* $\mathcal{C}$. ▲

The following is clear.

Lemma 5.30. Let $\mathcal{C}$ be a circle in an circle diagram $ST^{\star}$. Then the notions in Definition 5.29 are well-defined and satisfy:

(5.30.a) If $\mathcal{C}$ is usual & anticlockwise, then its interior is to the left. If $\mathcal{C}$ is usual & clockwise, then its exterior is to the left.

(5.30.b) If $\mathcal{C}$ is essential & leftwards, then its lower is to the left. If $\mathcal{C}$ is essential & rightwards, then its upper is to the left. ■

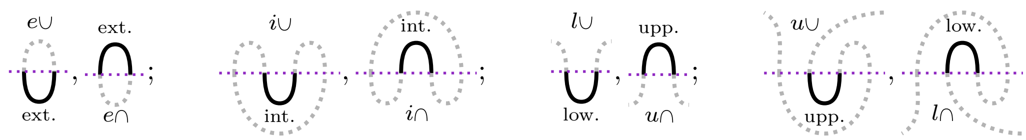
We also need to distinguish certain types of cups and caps.

Definition 5.31. Let $ST^{\star}$ be a circle diagram and $\mathcal{C}$ a circle in $ST^{\star}$.

(5.31.a) Let $\mathcal{C}$ be usual. We say that a cup, respectively cap, in $\mathcal{C}$ is $e\cup$, respectively $e\cap$, if the exterior of $\mathcal{C}$ is directly above the cup, respectively below the cap. Otherwise we call it $i\cup$, respectively $i\cap$.

(5.31.b) Let $\mathcal{C}$ be essential. We say that a cup, respectively cap, in $\mathcal{C}$ is $l\cup$, respectively $l\cap$, if the lower of $\mathcal{C}$ is directly below the cup, respectively below the cap. Otherwise we call it $u\cup$, respectively $u\cap$. ▲

Note that Definition 5.31 depends only on the shape, and here the picture:



We write e.g. $e\cup$ instead of $e\cup$ cup for short.

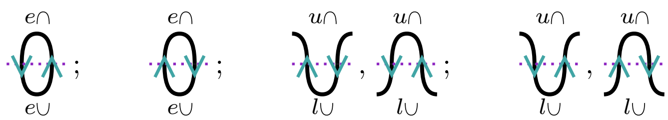
Lemma 5.32. Let $\mathcal{C}$ be a circle in an oriented circle diagram $\mathtt{C}^{\lambda}_{S,T}$.

(5.32.a) If $\mathcal{C}$ is usual, then the orientation of $\mathcal{C}$ and any $e\cup$ or $e\cap$ agrees, while any $i\cup$ or $i\cap$ is oriented in the opposite way.

(5.32.b) If $\mathcal{C}$ is essential & leftwards, then any $l\cup$ or $l\cap$ is oriented clockwise, while any $u\cup$ or $u\cap$ is oriented anticlockwise.

(5.32.c) If $\mathcal{C}$ is essential & rightwards, then any $u\cup$ or $u\cap$ is oriented clockwise, while any $l\cup$ or $l\cap$ is oriented anticlockwise. □

Proof. All of these are easily proved by induction on the number of cups and caps in the circle. Here the induction start:

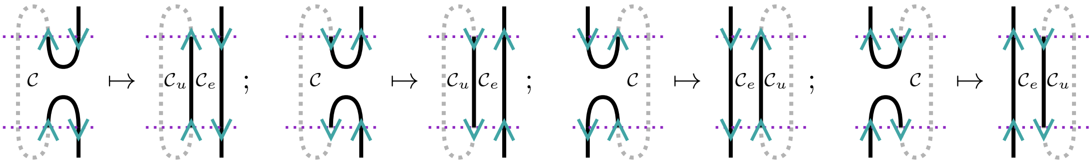


Then one continues using (5-zigzag). ■

For the next two lemmas the circles are considered inside an oriented, stacked diagram on which surgery is performed. Note hereby that we apply (5.surgery) only, i.e. without reorienting the resulting diagram, but rather keeping the original orientation. We call this *applying the surgery naively.*

Lemma 5.33. Assume an essential circle $\mathcal{C}$ splits into an essential $\mathcal{C}_e$ and an usual $\mathcal{C}_u$ circle by naive surgery. Then the resulting diagram is oriented, $\mathcal{C}_e$ is oriented in the same way as $\mathcal{C}$ and $\mathcal{C}_u$ is oriented opposite to the orientation of the $\cup$-$\cap$ pair involved in the naive surgery. □

Proof. First – by (5.32.b) and (5.32.c) – we know that the cup and cap involved in the naive surgery have the same orientation. Thus, these are the local possibilities:



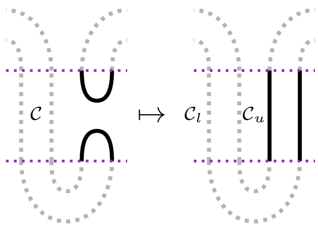
$\mathcal{C}_e$ is – by assumption – essential which means that all other possible situations can be rotated into such positions. ■

For two essential circles $\mathcal{C}_u$ is *above* $\mathcal{C}_l$ if $\mathcal{C}_l$ is contained in the lower half of $\mathcal{C}_u$. We also say that $\mathcal{C}_l$ is *below* $\mathcal{C}_u$.

Lemma 5.34. Let $\mathcal{C}$ be a usual circle splitting into two essential circles, $\mathcal{C}_u$ being above $\mathcal{C}_l$, by naive surgery. Then the result is oriented with $\mathcal{C}_u$ being essential & leftwards and $\mathcal{C}_l$ essential & rightwards in case $\mathcal{C}$ is anticlockwise, and vice versa, in case $\mathcal{C}$ is clockwise. □

Proof. As before by (5.32.a), we know that the $\cup$-$\cap$ of the naive surgery have the same orientation. Thus, there is an induced orientation on the result after naive surgery.

To see the second part of the claim, keeping



in mind, we use (5.30.a) and (5.30.b) with the interior of $\mathcal{C}$ turning into the lower of $\mathcal{C}_u$ and the upper of $\mathcal{C}_l$. ■

For the following lemma we use the evident notion of usual circles to be *nested* inside other usual circles. (We also say that one circle is *the outer* having the evident meaning.)

Lemma 5.35. Let $\mathtt{C}^{\lambda}_{S,T}$ be an oriented circle diagram.

(5.35.a) Let $\mathtt{C}^{\mu}_{S,T}$ be obtained from $\mathtt{C}^{\lambda}_{S,T}$ by reorienting an anticlockwise circle $\mathcal{C}$ clockwise, as well as reorienting an arbitrary number of clockwise circles nested inside $\mathcal{C}$ anticlockwise. Then $\mu <_{\varepsilon_S} \lambda$ and $\mu <_{\varepsilon_T} \lambda$.

(5.35.b) Assume that T is of type stay and let $\mathtt{C}^{\mu}_{S,T}$ be obtained from $\mathtt{C}^{\lambda}_{S,T}$ by reorienting a leftwards circle $\mathcal{C}$ rightwards, as well as reorienting an arbitrary number of rightwards circles below $\mathcal{C}$ leftwards. Then $\mu <_{\varepsilon_T} \lambda$. □

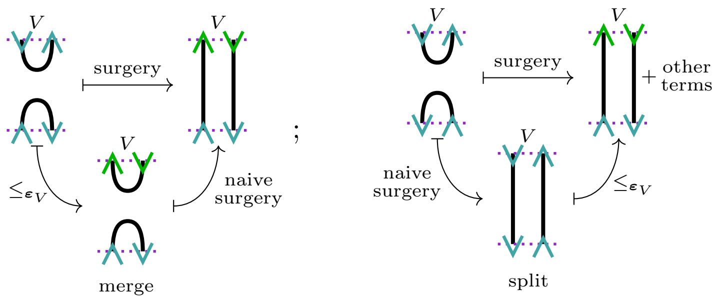
Proof. *(5.35.a).* We first use the rotation map ρ to obtain a diagram with S of type stay. Then the statement $\mu <_{\varepsilon_S} \lambda$ follows by the same arguments as in the usual case and is left to the reader. (For a similar proof see [ES16b, Lemma 7.7].) The same can be done to obtain a diagram with T of type stay giving $\mu <_{\varepsilon_T} \lambda$.

(5.35.b). In this case we substitute all cups in S that are not of type stay by cups of type stay that connect the same vertices to obtain a cup diagram S'. Then the circle $\mathcal{C}$ determines a circle $\mathcal{C}'$ in $S'T^{\star}$ containing the same caps as $\mathcal{C}$. Observe that $\mathcal{C}'$ is then anticlockwise. Hence, reorienting this we obtain – by (5.35.a) – a weight $\mu <_{\varepsilon_T} \lambda$. If there

are rightward circles below, they get transformed to clockwise circles nested inside $\mathcal{C}'$. So the statement also follows by (5.35.a), if some of these are reoriented. ∎

5I. Relative cellularity: Main proof. We can now proceed and finish with the proof of Theorem 5.16 to obtain the main part of relative cellularity for $\mathfrak{A}rc_n^{\mathrm{ann}}$.

Proof of Theorem 5.16. We show a stronger statement. Namely the appropriate analog of the claim itself, but for each step within the multiplication process. In each step the general idea is roughly as follows:



In words, we reorient before or after the surgery such that naive surgery gives the result we want to consider. In doing so the reordering will – by Lemma 5.35 – $\leq_{\varepsilon_V}$-decrease the weight. Observe hereby that this reorientation process is always possible. But in case of a merge the reorientation might happen for circles not touching the upper dotted line. (Examples are for instance provided by the merge rule (5.m1).) Those cases need a bit more care, but this will only happen in 5.16.Case.C below.

Let us make this rigorous. To this end, let $S\lambda T^{\star}T\mu V^{\star}$ be a stacked diagram. Without loss of generality we also assume that the diagram is rotated in such a way that V is of type stay. Further, let $\cap$ denote a cap in $T^{\star}$ and $\cup$ the mirrored cup in T such that one can perform surgery with the pair $\cup$-$\cap$. In the following, let $\mathcal{C}$ denote the circle containing $\cap$ and $\mathcal{C}'$ the circle containing $\cup$. (These need not be distinct in general.)

5.16.Claim.a. After naive surgery along $\cup$-$\cap$ and reorientation one obtains diagrams with an orientation μ' on the upper dotted line such that $\mu' <_{\varepsilon_V} \mu$. Further, if μ appears, then it appears with coefficient one, independent of V.

Proof of 5.16.Claim.a. The proof is divided into three parts: First we assume that $\cap$ is oriented clockwise, then we assume that $\cap$ is anticlockwise and divide the cases of $\cup$ being anticlockwise respectively clockwise. In all cases we silently use Lemma 5.32.

5.16.Case.A: $\cap$ is clockwise. We further distinguish depending on the properties of the circle $\mathcal{C}$, which in turn imply further properties of $\cap$ and $\cup$.

- ▷ *$\mathcal{C}$ is usual & anticlockwise.* This implies that $\cap$ is *i*$\cap$.
  - ▶ If the surgery is a merge of two circles, then $\mathcal{C}'$ must be nested inside $\mathcal{C}$. Hence, $\mathcal{C}'$ is usual as well, and $\mathcal{C}'$ and $\cup$ have the same orientation. In particular, if $\mathcal{C}'$ and $\cup$ are anticlockwise, we need to reorient $\mathcal{C}'$ and $\cup$ clockwise and then perform the surgery naively. The resulting orientation μ' on the upper dotted line is strictly $<_{\varepsilon_V}$-smaller than μ. If on the other hand $\mathcal{C}'$ and $\cup$ are clockwise, we need to reorient both $\mathcal{C}$ and $\mathcal{C}'$ and then perform the surgery naively. In this case this also produces a μ' strictly $<_{\varepsilon_V}$-smaller than μ.
  - ▶ If the surgery is a split, then $\cup$ is clockwise as well. Hence, the surgery will create two usual circles both containing arcs in V. Note that the naive surgery

creates two circles which are usual & anticlockwise. Thus, for each summand of the result one of the two circles needs to be reoriented creating strictly $<_{\boldsymbol{\varepsilon}_V}$-smaller orientations μ'.

▷ $\mathcal{C}$ *is usual & clockwise.* In this case $\cap$ is $e\cap$.

- ▶ If one merges, then the only non-zero result occurs when $\mathcal{C}'$ is usual & anticlockwise. To obtain the result we need to reorient $\mathcal{C}'$, and $\mathcal{C}$ if it is nested inside $\mathcal{C}'$, and then perform naive surgery. Since $\mathcal{C}'$ contains arcs in V this will produce a strictly $<_{\boldsymbol{\varepsilon}_V}$-smaller orientation μ'.
- ▶ If one splits, then $\cup$ is a clockwise $e\cup$. The only non-zero result is the split into two usual circles, both touching the upper dotted line. After performing naive surgery the outer of the two created circles is already clockwise, while the nested is anticlockwise. Reorienting the nested circle again gives a strictly $<_{\boldsymbol{\varepsilon}_V}$-smaller orientation μ'.

▷ $\mathcal{C}$ *is essential & leftwards.* In this case $\cap$ is $l\cap$.

- ▶ If the surgery is a merge, the non-zero cases are the ones where $\mathcal{C}'$ is usual & anticlockwise or essential & rightwards. In the first case, we have that $\cup$ is anticlockwise as well. In this case $\mathcal{C}'$ needs to be reoriented, strictly $<_{\boldsymbol{\varepsilon}_V}$-decreasing the orientation μ', and then naive surgery can be performed. In the second case, $\cup$ is clockwise. Performing naive surgery will then produce a usual & anticlockwise circle containing arcs in V. Thus, reorienting the resulting circle gives a $<_{\boldsymbol{\varepsilon}_V}$-strictly smaller orientation μ'.
- ▶ If the surgery is a split, then also $\cup$ is clockwise. Performing naive surgery will thus produce a usual & anticlockwise and an essential & leftwards circle. Since both contain arcs in V, reorienting the former will again yield a strictly $<_{\boldsymbol{\varepsilon}_V}$-smaller orientation μ' by Lemma 5.35.

▷ $\mathcal{C}$ *is essential & rightwards.* Very similar to the leftwards case and omitted.

5.16.Case.B: $\cap$ and $\cup$ are anticlockwise. In this case the result after naive surgery will always be automatically oriented, giving a coefficient 1 for the orientation μ. It remains to rule out the case that other summands are not $<_{\boldsymbol{\varepsilon}_V}$-strictly smaller than μ.

▷ We first assume that the surgery will be a merge. In case that two usual circles are merged, the result is already oriented in the correct way and no reorientation is necessary. In case that two essential circles are merged, note that either $\cap$ or $\cup$ is upper, while the other is lower. This means that one has an essential & leftwards and an essential & rightwards circle. Since the result of the naive surgery is oriented clockwise, no reorientation is needed. Last, if the merge includes a usual and an essential circle, then usual circle is oriented anticlockwise. Thus, there is again no need for a reorientation after surgery.

▷ Assume now that the surgery is a split. If it is a split into two usual circles, then original circle was anticlockwise. After naive surgery we get a usual & anticlockwise outer and a usual & clockwise nested circle. Thus, we obtain this as a summand in the result and a summand where both circles are reoriented. But since both contain arcs in V, this creates a strictly $<_{\boldsymbol{\varepsilon}_V}$-smaller orientation μ' on the upper dotted line. In case that the split creates a usual and an essential circle, then the usual circle is automatically anticlockwise after naive surgery. Finally, if the split creates two essential circles, then $\mathcal{C} = \mathcal{C}'$ is anticlockwise. Further, the upper of the two created circles is essential & leftwards, while the lower is essential & rightwards after naive surgery. The second summand in the result is obtained by reorienting both circles, but since both contain arcs in V, we see that reorienting both will give a strictly $<_{\varepsilon_V}$-smaller orientation μ'.

5.16.Case.C: $\cap$ is anticlockwise, $\cup$ is clockwise. This case is a bit different than the previous cases since the result will depend on whether the circle $\mathcal{C}$ contains arcs in V or not, and what we show is that the result will always be independent of V.

Before we start, we note that, since the orientations of $\cap$ and $\cup$ are different, the surgery will always be a merge.

- ▷ First assume that the circle $\mathcal{C}$ does not contain arcs in V. If $\mathcal{C}$ is usual, then a merge with an usual or essential circle $\mathcal{C}'$ will be performed by reorienting $\mathcal{C}$ followed by naive surgery. Hence, always resulting in the weight μ in the result. In case $\mathcal{C}$ is essential, the two possibilities for $\mathcal{C}'$ are either an essential circle, which would be oriented in the same way as $\mathcal{C}$, or $\mathcal{C}'$ being usual & clockwise. Both cases result in zero. Thus, in this case the result is independent of V.
- ▷ If on the other hand $\mathcal{C}$ contains arcs in V, then $\mathcal{C}$ being usual will always strictly $<_{\varepsilon_V}$-decrease the weight μ' when $\mathcal{C}$ is reoriented. While the case $\mathcal{C}$ being essential, would still result in zero in all cases. Since in this case μ never occurs, its coefficient is again independent of V.

In the last case, the condition whether $\mathcal{C}$ contains arcs in V or not is equivalent to asking whether swapping all entries in λ contained in the circle $\mathcal{C}$ would give an orientation of $\mathcal{C}$ or not. If $\mathcal{C}$ does not contain arcs in V then it would just be the opposite orientation, while if $\mathcal{C}$ contains arc in V, this would not result in an orientation as the orientation on the top is unchanged. Doing this for all surgery moves and always assuming the case that λ appears in every step, thus implies that $V \in \mathsf{M}(\lambda)$.

Taking all above together shows 5.16.Claim.a which in turn implies the statement. ■

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